ALTERNATIVE DERIVATION OF SOME REGULAR CONTINUED FRACTIONS

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In this paper we find an expression for e^x as the limit of quotients associated with a sequence of matrices, and thence, by using the matrix approach to continued fractions ([5] 12—13, [2] and [4]), we derive the regular continued fraction expansions of $e^{2/k}$ and $\tan 1/k$ (where k is a positive integer).

If the real number α has the regular continued fraction expansion

$$\alpha = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots$$

(which in this paper we shall write as

$$\alpha = [a_0, a_1, a_2, \cdots]),$$

then it is easy to see that the convergents p_n/q_n to α are given by

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus α may be expressed as a limit of quotients, namely p_n/q_n, associated with the sequence of matrices $\left\{\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\right\}$.

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We now introduce some notation. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix, for definiteness over the field of complex numbers, we define

$$K_1(A) = \frac{a}{c}, \text{ if } c \neq 0; \quad K_2(A) = \frac{b}{d}, \text{ if } d \neq 0.$$

If $\{A_n\}$ is a sequence of such matrices, and

$$K_s(A_1 \cdots A_n) \to \alpha_s \qquad (s = 1, 2)$$

as $n \to \infty$, we say that $K_s(A_1 \cdots A_n)$ converges to α_s and we write

$$K_s(A_1A_2\cdots) = \alpha_s.$$

If $\alpha_1 = \alpha_2 = \alpha$, we write simply

$$K(A_1A_2\cdots) = \alpha.$$

With this notation,

$$[a_0, a_1, a_2, \cdots] = K\left\{\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\right\}. \tag{2.1}$$

Thus, if $a_0, a_1, a_2, \cdots$ is a sequence of integers, all positive except (perhaps) a_0, then the right-hand side of (2.1) represents a unique real number α and the regular continued fraction expansion of this number is

$$\alpha = [a_0, a_1, a_2, \cdots].$$

We note here some simple properties of the functions K_1, K_2. Lemmas 1 to 3 are stated in terms of K_1, but apply equally to K_2.

LEMMA 1. *If* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $K_1(A_1A_2\cdots) = \alpha$, *where* $c\alpha+d \neq 0$, *then*

$$K_1(BA_1A_2\cdots) = \frac{a\alpha+b}{c\alpha+d}.$$

LEMMA 2. *If* $K_1(A_1A_2\cdots) = \alpha$, *and* $\{k_n\}$ *is a sequence of non-zero complex numbers, then*

$$K_1\{(k_1A_1)(k_2A_2)\cdots\} = \alpha.$$

LEMMA 3. *Suppose* $K_1(A_1A_2\cdots)$ *exists. If* $\{B_1B_2\cdots B_n\}$ *is a subsequence of* $\{A_1A_2\cdots A_n\}$, *then*

$$K_1(A_1A_2\cdots) = K_1(B_1B_2\cdots);$$

in particular

$$K_1(A_1A_2\cdots) = K_1\{(A_1A_2A_3)(A_4A_5A_6)\cdots(A_{3n-2}A_{3n-1}A_{3n})\cdots\}.$$

LEMMA 4. *If* $A_1\cdots A_n = \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix} = P_n$, *then*

$$|K_1(P_n)-K_2(P_n)| = \frac{1}{|q_ns_n|}\prod_{r=1}^{n}|\det A_r|.$$

The proofs of Lemmas 1 to 4 are trivial and are left to the reader.

LEMMA 5. *Let* $B, A_1, A_2, \cdots$ *be matrices over the ring of Gaussian integers, with* $|\det A_r| = 1$ $(r = 1, 2, \cdots)$ *and* $K(A_1A_2\cdots) = \alpha$. *Then, if*

$$B \neq \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix},\ K_1(A_1\cdots A_nB) \to \alpha \text{ as } n \to \infty;$$

and if

$$B \neq \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix},\ K_2(A_1\cdots A_nB) \to \alpha.$$

If B *has a non-zero element in each column,* $K(A_1\cdots A_nB) \to \alpha$.

PROOF. It suffices to prove the first conclusion of the lemma; the proof of the second is similar, and the final result follows immediately. We write

$$A_1A_2\cdots A_n = \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\alpha_n = \frac{ap_n+cr_n}{aq_n+cs_n} = K_1(A_1\cdots A_nB).$$

If a or c is zero, the result is trivial, so we suppose that neither is zero.

Since $K(A_1A_2\cdots)$ exists, Lemma 4 shows that $|q_ns_n| \to \infty$ as $n \to \infty$. Also $(q_n, s_n) = 1$, because

$$|p_ns_n - q_nr_n| = |\det(A_1A_2\cdots A_n)| = 1.$$

Hence $|aq_n+cs_n| \geqq 1$ for all large n, since aq_n+cs_n is a Gaussian integer and $aq_n+cs_n = 0$ implies q_ns_n divides ac, which is impossible for sufficiently large n. Thus

$$\left|\alpha_n - \frac{p_n}{q_n}\right|\left|\alpha_n - \frac{r_n}{s_n}\right| = \frac{|ac|}{|q_ns_n|\,|aq_n+cs_n|^2} \to 0$$

as $n \to \infty$. Since p_n/q_n and r_n/s_n both tend to α as $n \to \infty$, it now follows that $\alpha_n \to \alpha$.

The next result is of fundamental importance.

LEMMA 6. *Let $B, A_1, A_2, \cdots$ be non-singular matrices over the ring of Gaussian integers, with*

$$|\det A_r| = 1 \ (r = 1, 2, \cdots) \text{ and } BC_rB^{-1} = A_r \ (r = 1, 2, \cdots).$$

Then $K(A_1A_2\cdots) = \alpha$ implies $K(BC_1C_2\cdots) = \alpha$.

PROOF. From Lemma 5, $K(A_1\cdots A_nB) \to \alpha$ as $n \to \infty$. Noting that $BC_1\cdots C_n = A_1\cdots A_nB$ we have $K(BC_1\cdots C_n) \to \alpha$ as $n \to \infty$, the result.

It is of particular interest to evaluate $K(A_1A_2\cdots)$ in the case where A_r has rational integral elements with $|\det A_r| = 1$, since it will then frequently be possible to transform the product into one of the form exhibited in (2.1), so yielding a regular continued fraction. A useful result in this connection is

LEMMA 7. *If a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has integral elements with $c > d > 0$ and determinant ± 1, then*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

where a_0 is an integer and $a_1, \cdots, a_n$ are positive integers.

PROOF. If $d > 1$, we write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & a-xb \\ d & c-xd \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},$$

where $x = [c/d] \geqq 1$. Noting that c/d is not an integer, since $d|c$ would imply $d|1$, we have $d > c-xd > 0$. Repetition of this process must lead ultimately to the case $d = 1$.

If $d = 1$, then $c > 1$ and $a-bc = \pm 1$, and we have

$$\begin{pmatrix} a & b \\ c & 1 \end{pmatrix} = \begin{pmatrix} b & a-bc \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a-(c-1)b & bc-a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c-1 & 1 \\ 1 & 0 \end{pmatrix},$$

and one of these products is of the required form.

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We now obtain an expression for e^x in the form $K(A_1A_2A_3\cdots)$.

THEOREM 1.

$$e^x = K\left\{\prod_{m=0}^{\infty} \begin{pmatrix} (2m+1)+x & (2m+1) \\ (2m+1) & (2m+1)-x \end{pmatrix}\right\} \tag{3.1}$$

for all (complex) x.

PROOF. We first show that

$$\prod_{m=1}^{n} \begin{pmatrix} (2m-1)+x & (2m-1) \\ (2m-1) & (2m-1)-x \end{pmatrix} = \begin{pmatrix} f_n(x) & g_n(x) \\ h_n(x) & k_n(x) \end{pmatrix}, \tag{3.2}$$

where

$$h_n(x) = g_n(-x), \quad k_n(x) = f_n(-x) \tag{3.3}$$

and

$$f_n(x) = \sum_{k=0}^{n} nc_{n,k}x^k,$$

$$g_n(x) = \sum_{k=0}^{n} (n-k)c_{n,k}x^k,$$

with

$$c_{n,k} = \frac{(2n-k-1)!}{(n-k)!k!}.$$

The relations (3.3) follow immediately from the observation that the left-hand side of (3.2) is unchanged on interchanging rows and then columns of each of the matrices and replacing x by $-x$.

We now proceed by induction on n. The result is clearly true for $n = 1$, and we assume it true for some $n \geqq 1$. To prove it true for $n+1$ it suffices, in view of (3.3), to show that

$$
\begin{aligned}
(3.4) \qquad & (2n+1)\{f_n(x)+g_n(x)\}+xf_n(x) = f_{n+1}(x), \\
& (2n+1)\{f_n(x)+g_n(x)\}-xg_n(x) = g_{n+1}(x),
\end{aligned}
$$

and it is easily verified that these relations hold.

To establish (3.1), then, we must prove that $f_n(x)/g_n(-x) \to e^x$ as $n \to \infty$. This follows from the results

$$\frac{f_n(x)}{n(n+1)\cdots(2n-1)} \to e^{\frac{1}{2}x}, \quad \frac{g_n(x)}{n(n+1)\cdots(2n-1)} \to e^{\frac{1}{2}x}.$$

We prove the first of these. Using the expression for $f_n(x)$, we have for all complex x

$$
\begin{aligned}
\frac{f_n(x)}{n(n+1)\cdots(2n-1)} &= 1+\sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{(2n-1)(2n-2)\cdots(2n-k)}\frac{x^k}{k!} \\
&= 1+\sum_{k=1}^{n} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{\left(1-\frac{1}{2n}\right)\left(1-\frac{2}{2n}\right)\cdots\left(1-\frac{k}{2n}\right)}\frac{(\frac{1}{2}x)^k}{k!} \\
&= 1+\sum_{k=1}^{n} a_{n,k}\frac{(\frac{1}{2}x)^k}{k!}, \text{ say.}
\end{aligned}
$$

Clearly $a_{n,k} \to 1$ as $n \to \infty$ for fixed k, and also

$$a_{n,k} < \frac{1}{\left(1-\frac{k}{2n}\right)^k} \leqq \frac{1}{(1-\frac{1}{2})^k} = 2^k,$$

so the first result stated follows from Tannery's theorem [1]. The second follows from the first and the relation (3.4) (if we divide by $(n+1)(n+2)\cdots(2n+1)$ and let $n \to \infty$).

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We now deduce some regular continued fractions from the relation (3.1). We collect them together in

THEOREM 2. *The following are regular continued fraction expansions for the functions specified, where k denotes an integer subject to the restrictions stated.*

(i) $e^{1/k} = [\overline{1,\ (2n+1)k-1,\ 1}]_{n=0}^{\infty}$ $(k > 1)$;

$e \ = [2, \overline{1,\ 2n,\ 1}]_{n=1}^{\infty}$.

(ii) $e^{2/k} = [\overline{1,\ \tfrac{1}{2}\{(6n+1)k-1\},\ 6(2n+1)k,\ \tfrac{1}{2}\{(6n+5)k-1\},\ 1}]_{n=0}^{\infty}$ (*odd* $k>1$);

$e^2 \ = [7, \overline{3n+2,\ 1,\ 1,\ 3n+3,\ 6(2n+3)}]_{n=0}^{\infty}$.

(iii) $\tan\dfrac{1}{k} = [0, k-1, \overline{1,\ (2n+1)k-2}]_{n=1}^{\infty}$ $(k > 1)$;

$\tan 1 = [\overline{1,\ 2n-1}]_{n=1}^{\infty}$.

(The beginning of the expansion of e^2 illustrates the meaning of the notation we have used:

$$e^2 = [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, \cdots]).$$

Proof of Theorem 2. (i) If we put $x = 1/k$ in (3.1), with integral $k > 0$ and use Lemma 2, we obtain

$$e^{1/k} = K\left\{\prod_{n=0}^{\infty}\begin{pmatrix}(2n+1)k+1 & (2n+1)k \\ (2n+1)k & (2n+1)k-1\end{pmatrix}\right\}.$$

Using the result

$$\begin{pmatrix}a+1 & a \\ a & a-1\end{pmatrix} = \begin{pmatrix}1 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}a-1 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 \\ 1 & 0\end{pmatrix} \tag{4.1}$$

with $a = (2n+1)k$, we find

$$e^{1/k} = [\overline{1,\ (2n+1)k-1,\ 1}]_{n=0}^{\infty},$$

which is a regular continued fraction if $k > 1$.

If $k = 1$, on using (4.1) with $a = 2n+1$ and $n > 0$, we obtain

$$e = [2, \overline{1,\ 2n,\ 1}]_{n=1}^{\infty}.$$

(ii) Similarly

$$e^{2/k} = K\left\{\prod_{\nu=0}^{\infty}\begin{pmatrix}(2\nu+1)k+2 & (2\nu+1)k \\ (2\nu+1)k & (2\nu+1)k-2\end{pmatrix}\right\}.$$

We may transform the product of three successive factors in this expression, given by $\nu = 3n,\ 3n+1,\ 3n+2$, into a form which is appropriate when k is an odd integer. We observe that

$$\begin{aligned}\begin{pmatrix}a+2 & a \\ a & a-2\end{pmatrix} &= \begin{pmatrix}1 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\tfrac{1}{2}(a-1) & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 2 \\ 1 & -1\end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix}1 & 2 \\ 1 & -2\end{pmatrix}\begin{pmatrix}2a & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 \\ 2 & -2\end{pmatrix} \\ &= \begin{pmatrix}2 & 1 \\ 2 & -1\end{pmatrix}\begin{pmatrix}\tfrac{1}{2}(a-1) & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 \\ 1 & 0\end{pmatrix},\end{aligned}$$

and apply these factorizations in this order to the three matrices specified, noting that

$$\begin{pmatrix}2 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & -2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}2 & 1\\ 2 & -1\end{pmatrix} = 4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

This yields

$$\prod_{r=1}^{3}\begin{pmatrix}(6n+2r-1)k+2 & (6n+2r-1)k\\ (6n+2r-1)k & (6n+2r-1)k-2\end{pmatrix}$$

$$= 8\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\{(6n+1)k-1\} & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}6(2n+1)k & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\{(6n+5)k-1\} & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},$$

and hence the results stated. (For other methods of establishing this result, and the one for $e^{1/k}$ see [5], 123—125, or [3].)

(iii) Since

$$\cot\frac{1}{k}-1 = \frac{(i-1)e^{2i/k}+(i+1)}{e^{2i/k}-1},$$

application of Lemma 1 to (3.1) with $x = 2i/k$ gives

$$\cot\frac{1}{k}-1 = K\left\{\begin{pmatrix}i-1 & i+1\\ 1 & -1\end{pmatrix}\prod_{n=0}^{\infty}\begin{pmatrix}(2n+1)k+2i & (2n+1)k\\ (2n+1)k & (2n+1)k-2i\end{pmatrix}\right\}.$$

Since

$$\begin{pmatrix}i-1 & i+1\\ 1 & -1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}a+2i & a\\ a & a-2i\end{pmatrix} = \begin{pmatrix}a-1 & a-2\\ 1 & 1\end{pmatrix}\begin{pmatrix}i-1 & i+1\\ 1 & -1\end{pmatrix},$$

and

$$\begin{pmatrix}a-1 & a-2\\ 1 & 1\end{pmatrix} = \begin{pmatrix}a-2 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},$$

Lemma 6 leads to the result

$$\cot\frac{1}{k}-1 = [\overline{(2n+1)k-2,\ 1}]_{n=0}^{\infty}.$$

It follows that

$$\cot\frac{1}{k} = [k-1, \overline{1,\ (2n+1)k-2}]_{n=1}^{\infty},$$

and so

$$\tan\frac{1}{k} = [0, k-1, \overline{1,\ (2n+1)k-2}]_{n=1}^{\infty};$$

these are regular continued fractions for integers $k > 0$ and $k > 1$, respectively.

Finally,

$$\tan 1 = [\overline{1, 2n-1}]_{n=1}^{\infty}.$$

Notice that these expansions for $\tan 1/k$ can be derived from Lambert's semiregular continued fraction expansion for $\tan 1/k$ ([5], 148–149 and [6]).

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