A really simple proof of the Markoff conjecture for prime powers

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A triple (a, b, c) of positive integers that satisfies the diophantine equation

$$a^2 + b^2 + c^2 = 3abc$$

is called a Markoff triple and the numbers a, b and c are called Markoff numbers. The Markoff conjecture states that if c is a Markoff number and (a_1, b_1, c) and (a_2, b_2, c) are two Markoff triples with $a_1 \leq b_1 \leq c$ and $a_2 \leq b_2 \leq c$, then $a_1 = a_2$ and $b_1 = b_2$.

Given any Markoff triple (a, b, c), one can obtain three more triples, namely, $(a, b, 3ab - c), (a, c, 3ac - b)$ and $(b, c, 3bc - a)$, which are called neighbours of (a, b, c). In [1] Markoff showed that starting from $(1, 1, 1)$, one can obtain all Markoff triples by generating these neighbours at each step. As a consequence of this result it follows easily that the three elements of a Markoff triple are pairwise coprime.

Lemma Let (a_1, b_1, c) and (a_2, b_2, c) be two Markoff triples. Assume that $(a_2, b_2) \neq (a_1, b_1)$ or (b_1, a_1). Then there exist coprime integers p and q with $c = pq$ such that $a_1a_2 - b_1b_2 \equiv 0 \mod p^2$ and $a_1b_2 - b_1a_2 \equiv 0 \mod q^2$.

Proof We have

$$(a_1a_2 - b_1b_2)(a_1b_2 - b_1a_2) = c^2(a_1b_1 - a_2b_2).$$

Assume that $a_1b_1 - a_2b_2 = 0$. Then either $a_1a_2 = b_1b_2$ or $a_1b_2 = b_1a_2$. Let $a_1a_2 = b_1b_2$. As $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$ we have $(a_2, b_2) = (b_1, a_1)$. Similarly if $a_1b_2 = b_1a_2$, we have $(a_2, b_2) = (a_1, b_1)$. Hence $a_1b_1 - a_2b_2 \neq 0$.

Let $g > 1$ be a divisor of c. We will show that g cannot divide both $a_1a_2 - b_1b_2$ and $a_1b_2 - b_1a_2$. Assume on the contrary that $a_1a_2 \equiv b_1b_2 \mod g$. and $a_1b_2 \equiv b_1a_2 \mod g$. On multiplication of the two congruences we obtain

$a_1^2 a_2 b_2 \equiv b_1^2 a_2 b_2 \mod g$. It follows that $a_1^2 \equiv b_1^2 \mod g$ as $\gcd(a_2 b_2, c) = 1$. However as $g | a_1^2 + b_1^2$ we have $g | b_1$ which is not true as $\gcd(c, b_1) = 1$. Therefore $\gcd(a_1 a_2 - b_1 b_2, a_1 b_2 - b_1 a_2, c) = 1$ and the lemma follows.

Theorem The Markoff conjecture is true for a Markoff number c that is a prime power.

Proof Let (a_1, b_1, c) and (a_2, b_2, c) be two Markoff triples with $a_i \leq b_i \leq c$ for $i = 1, 2$. If $(a_2, b_2) = (b_1, a_1)$, then $a_1 = b_1 = a_2 = b_2 = 1$. Otherwise if the two triples are distinct, then by Lemma there exist coprime integers p, q with $c = pq$ such that $a_1 a_2 - b_1 b_2 \equiv 0 \mod p^2$ and $a_1 b_2 - b_1 a_2 \equiv 0 \mod q^2$. As c is a prime power, we conclude that one of p or q say q, is equal to 1. Then $p = c$ and $a_1 a_2 - b_1 b_2 \equiv 0 \mod c^2$. As $a_i, b_i \leq c$, it follows that $a_1 a_2 - b_1 b_2 = 0$ which, as seen in the proof of the lemma implies that $(a_2, b_2) = (b_1, a_1)$, which has been ruled out.

References

[1] A. A. Markoff, *Sur les formes quadratiques binaires indefiniesi, I*, Math. Ann. **15**. (1879), 381-409