

ANDREI BORISOVICH SHIDLOVSKY (OBITUARY)

Andrei Borisovich Shidlovsky, Honoured Scientist of the Russian Federation and Honoured Professor of Moscow State University, passed away in his 91st year of life on 23 March 2007. He was a great Russian mathematician, one of the leading number theorists in Russia.

Shidlovsky [Shidlovskii] was born in 1915 in the town Alatyr' in the Chuvash Republic. His father, Boris Andreevich Shidlovsky, came from an impoverished noble family from the Simbirsk Province; after graduating from the St Petersburg Polytechnic Institute he worked as a land-tenure regulator, was an elected presiding magistrate, a member of the Bar, and worked as an economist. His mother, Aleksandra Vsevolodovna Shidlovskaya, born Skorokhodova, completed her education at the Simbirsk gymnasium and was employed for many years as a typist in various organizations.

Shidlovsky trod a long and complicated career path. In 1930 he completed his seven years of school in Ul'yanovsk. Then the family moved to Moscow, and after attending a school for apprentices he worked for several years as a lathe operator. During his study and afterwards he supplemented his income by playing in, and indeed leading, a wind orchestra. In 1934, responding to Komsomol propaganda, he went to take part in the construction of the first line of the Moscow underground, where he worked as a tunneller. After his military service in the Soviet Army in 1936–37 he completed evening school, and in 1939 he entered the Faculty of Mechanics and Mathematics of Moscow State University. As soon as the Great War for the Motherland began, on 26 June 1941, he joined the army as a volunteer; he began as a junior topographer at the Bryansk front, and finished as the commander of a self-propelled artillery unit in China. In 1942 he became a probationary member, and in 1944 a full member, of the Communist Party of the Soviet Union.

From demobilization in 1946 until 1950, he continued his interrupted study in the Faculty of Mechanics and Mathematics. In 1953 he finished the work for his Ph.D. and in 1954 he defended his Ph.D. dissertation. His research supervisor was Alexander Osipovich Gel'fond. While an undergraduate student and a Ph.D. student, Shidlovsky, already father of three daughters, taught at an evening school, and gave lectures and practical seminars at a technical institute. After finishing his Ph.D. he worked at the Moscow Pedagogical Institute as a senior lecturer, and later as a docent and part-time professor. In 1954, on the initiative of A. Ya. Khinchin, he was offered a position in the Department of Mechanics and Mathematics at Moscow State University, with which he was associated until the last days of his life. In 1955 he became a docent in the mathematical analysis group. In 1959 he defended his doctoral dissertation, and from 1960 worked as a professor in the number theory group. From 1968, after the death of Gel'fond, until 2002, he was the head of the number theory group. During his many years of work in the Faculty of Mechanics and Mathematics he taught the core courses in mathematical analysis and number

theory, and special courses in number theory, and he ran research seminars. Of his fifteen Ph.D. students three defended doctoral dissertations.

For several years Shidlovsky was deputy head of the Faculty of Mechanics and Mathematics, and he was a member of the faculty Party committee and later its leader. For a long time he was a member of the academic council of the Faculty of Mechanics and Mathematics, a member of the expert committee of the Higher Certification Commission of the USSR, and a member of the Moscow Mathematical Society. He has been awarded three orders and many medals.

Shidlovsky's research results are in transcendental number theory, whose main problems concern the irrationality and transcendence of various numbers, and, more generally, the proof that there are no algebraic relations among them over the rational field \mathbb{Q} . In 1873, C. Hermite proved that the number e is transcendental, that is, it is not a root of any non-zero polynomial with rational coefficients. In 1882, F. Lindemann proved more generally that for each algebraic number $\alpha \neq 0$ the exponential e^α is transcendental. Therefore, for $\alpha = \pi i$, it follows that π is transcendental. Another classical example is Hilbert's seventh problem, on the transcendence of numbers of the type α^β for algebraic numbers $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$. This problem was solved in 1934 by Gel'fond and independently by T. Schneider. This result implies in particular that the number $e^\pi = (-1)^{-i}$ is transcendental.

Shidlovsky proved his first results, on the quantitative transcendence of numbers of the form α^β , while he was still a student working under the direction of Gel'fond, and thereafter he devoted himself entirely to problems connected with the generalization of the results of Hermite and Lindemann.

Attempts to generalize these results were of course made immediately after their proofs. The function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

belongs to the class of so-called confluent hypergeometric functions

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{z^n}{n!}, \quad p \leq q. \quad (1)$$

Here, $(a)_0 = 1$ and $(a)_n = a(a+1) \cdots (a+n-1)$ for $n \geq 1$. Partial results about arithmetic properties of the values of such functions for rational z and rational values of the parameters a_j, b_j were obtained by A. Legendre, A. Hurwitz, E. Stridsberg, and W. Maier. In 1929, C. L. Siegel isolated properties that are relevant for the study of the transcendence of the values of such functions (algebraicity of the coefficients of the Taylor series, the rate of growth of these coefficients and their common denominators, and the existence of suitable linear differential equations) and took them as defining properties of a general class of functions that he called E -functions. The function e^z from which these functions derive their name, and the functions (1) for rational a_j, b_j , belong to the class of E -functions. Siegel proposed an approach to the investigation of the algebraic independence of values of such functions. However, he himself could only obtain complete results for functions of the type (1) with $q = 1$.

Siegel's programme was carried out in its entirety in a series of papers by Shidlovsky: he succeeded in proving some fundamental theorems which completely solved the problems of transcendence and algebraic independence of the values at algebraic points of E -functions.

The proof of algebraic independence of solutions of linear differential equations is a very complicated problem that remains unsolved in many cases. Nevertheless, in principle, Shidlovsky's theorem from 1954 completes the number-theoretic part of the investigation of arithmetic properties of values of E -functions at algebraic points, because it reduces in general an arithmetic problem to an analytic problem, namely, the proof of algebraic independence of functions.

Using the theorem, Shidlovsky obtained the first results on the transcendence and algebraic independence of values of the hypergeometric functions (1) satisfying differential equations of arbitrary orders, and he extended Siegel's ideas to E -functions satisfying non-homogeneous linear differential equations, and also to meromorphic functions that are quotients of E -functions. The simplicity of the conditions of the theorem allowed him to obtain various specific applications. One of the examples concerns the functions

$$\psi_k(z) = \sum_{n=0}^{\infty} \frac{z^{kn}}{(n!)^k}, \quad k \geq 1;$$

for each algebraic number $\alpha \neq 0$, any finite set of numbers of the form

$$\psi_k^{(l)}(\alpha), \quad k \geq 1, \quad 0 \leq l < k,$$

is algebraically independent.

Later on, Shidlovsky generalized his 1954 theorem to algebraically dependent E -functions. He was able to prove in general the equality of transcendence degrees of the functions over $\mathbb{C}(z)$ and of their values over \mathbb{Q} at an algebraic point (different from 0 and singular points of the corresponding system of differential equations); in a series of papers he obtained sufficient conditions establishing the algebraic independence of $f_1(\alpha), \dots, f_l(\alpha)$, $l < m$, for E -functions $f_1(z), \dots, f_m(z)$ such that $f_1(z), \dots, f_l(z)$ are algebraically independent over $\mathbb{C}(z)$.

Shidlovsky's theorems made it possible to obtain quantitative statements about algebraic independence of values of E -functions in terms of estimates of so-called algebraic independence measures. These results were obtained by S. Lang, Shidlovsky himself and A. I. Galochkin; they generalized the corresponding estimates obtained by Siegel. Shidlovsky succeeded in obtaining practically best-possible quantitative estimates in the case of E -functions when the coefficients of the Taylor series are from the rational field or an imaginary quadratic field.

Shidlovsky's general theorems stimulated the development of methods for proving the algebraic independence of solutions of linear differential equations over $\mathbb{C}(z)$. Shidlovsky and his students (I. I. Belogrivov, V. A. Oleinikov, Yu. V. Nesterenko, V. Kh. Salikhov and others) devoted a number of papers to this topic. In connection with applications of these ideas in transcendental number theory, the algebraic independence of solutions of linear differential equations was studied by K. Mahler, K. Väinänen, F. Beukers, W. D. Brownawell, and G. Heckman. Work has also been done on the determination of Galois groups of hypergeometric differential equations (Beukers).

The proof of Shidlovsky's theorems depended on sharp estimates for the multiplicity of a zero at a fixed point for the linear form

$$R(z) = P_0(z) + P_1(z)f_1(z) + \dots + P_m(z)f_m(z), \quad P_j(z) \in \mathbb{C}[z], \quad (2)$$

in terms of the degrees of the polynomials $P_j(z)$. This approach has exerted a strong influence on the development of transcendental number theory. Applied to

functions satisfying algebraic differential equations, it led to results on the transcendence measure of values of elliptic functions (Brownawell and D. Masser) and to progress on the arithmetic properties of values of modular forms. In particular, these ideas seem to be crucial in the proof of the algebraic independence of π and e^π (Nesterenko). Significant progress in obtaining estimates of the transcendence degree of fields generated by values of exponential and elliptic functions was made possible using estimates of the number of zeros of polynomials on algebraic groups (P. Philippon, M. Waldschmidt, G. Diaz). The best current estimates for linear forms in logarithms of algebraic numbers were also obtained using estimates of the number of zeros of polynomials on algebraic groups (A. Baker, G. Wüstholtz, E. M. Matveev). Different approaches have been developed for estimating the multiplicity of a zero of the linear form (2) (D. Bertrand, Beukers).

Shidlovsky's ideas have been used in the study of other classes of functions, and in particular the class of so-called G -functions, introduced by Siegel in 1929. Typical examples of G -functions are hypergeometric functions with rational parameters and finite radius of convergence (M. S. Nurmagomedov, Galochkin, G. V. Chudnovsky).

Shidlovsky's results are well known to experts both at home and abroad. His books have been published in translation, and he has travelled abroad extensively and taken part in various international conferences in number theory.

Intensive study of the arithmetic properties of values of E -functions is still continuing today. In 2000 Bertrand published a new proof of Shidlovsky's 1954 theorem, using the so-called interpolation determinants. In 2004 Beukers proved an old conjecture of Siegel on the linear independence of values of E -functions. One can confidently declare that the ideas in transcendental number theory introduced by Andrei Borisovich Shidlovsky and the theory that he established are still being successfully developed by mathematicians in Russia and abroad.

Andrei Borisovich Shidlovsky was an exceptionally noble, intelligent, and steadfast man who was severe but equitable to all. A bright memory of him will always be preserved in the hearts of those who knew him.