

# A unimodular matrix connected with Pell's equation

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**Theorem.** Suppose  $u^2 - Dv^2 = \epsilon = \pm 1$ , where  $u, v, D > 1$  and  $D$  is not a square. Then the unimodular matrix  $A = \begin{pmatrix} Dv & u \\ u & v \end{pmatrix}$  has the (unique) factorisation

$$A = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

where all  $a_i > 0$ . Also

- (a) the sequence  $a_0, \dots, a_n$  is palindromic,
- (b)  $\sqrt{D} = [a_0, \overline{a_1, \dots, 2a_0}]$ ,
- (c)  $u/v = [a_0, \dots, a_{n-1}]$ ,
- (d)  $Dv/u = [a_0, \dots, a_n]$ .

## Remarks

1. The period is a multiple of the least period and will be the least period if and only if  $u + v\sqrt{D}$  is the fundamental solution of  $u^2 - Dv^2 = \epsilon$ .
2. These results are also treated in [1, pages 110-111]. Also see [3, pages 276-281]. Our treatment is partly based on that in [2, §21].

**Lemma.** ([2, page 193].) Let  $a_1, \dots, a_n, x, y$  be positive real numbers,  $a_0$  a real number. Also let

$$z = [a_0, a_1, \dots, a_n, x/y].$$

Then  $z = r/s$ , where

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

**Proof** of the Theorem. We have

$$A \begin{pmatrix} 1 \\ \sqrt{D} \end{pmatrix} = \begin{pmatrix} Dv + u\sqrt{D} \\ u + v\sqrt{D} \end{pmatrix} = \begin{pmatrix} \sqrt{D}(u + v\sqrt{D}) \\ u + v\sqrt{D} \end{pmatrix}.$$

With  $x = 1, y = \sqrt{D}, r = \sqrt{D}(u + v\sqrt{D}), s = u + v\sqrt{D}$ , the Lemma then gives

$$\begin{aligned} \sqrt{D} &= [a_0, \dots, a_n, 1/\sqrt{D}] \\ &= [a_0, \dots, a_{n-1}, a_n + \sqrt{D}] \\ a_0 + \sqrt{D} &= [2a_0, \dots, a_{n-1}, a_0 + \sqrt{D}] \\ &= [2a_0, \dots, a_{n-1}] \\ \sqrt{D} &= [a_0, \overline{a_1, \dots, a_{n-1}}, 2a_0]. \end{aligned}$$

**Example.**  $D = 14, u = 15, v = 4$ . Then  $u^2 - Dv^2 = 1$  and

$$\begin{pmatrix} 56 & 15 \\ 15 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence

$$\begin{aligned} \sqrt{14} &= [3, \overline{1, 2, 1, 6}] \\ 56/15 &= [3, 1, 2, 1, 3] \\ 15/4 &= [3, 1, 2, 1]. \end{aligned}$$

## References

- [1] A.J van der Poorten, *An introduction to continued fractions*, LMS Lecture Note Series **109**, 99-138, CUP 1985
- [2] O. Forster, *Algorithmische Zahlentheorie*, Vieweg 1996
- [3] H. Koch, *Zahlentheorie: Algebraische Zahlen und Funktionen*, Vieweg 1997