Some continued fraction identities

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Abstract

Abstract. We generalise some Pell equation identities of Pohst and Zassenhaus [1, pp. 143] which enable one to use only half the period of the continued fraction expansion fo $\sqrt{D}$ when solving $x^2 - Dy^2 = N$ for $|N| < \sqrt{D}$.

In a similar vein, we also give a proof of identity (6) on page 104 of *Kettenbrüche* by Oscar Perron.

The simple continued fraction expansion for $\sqrt{D}$ is periodic with period k:

$$\sqrt{D} = \begin{cases} [a_0, \overline{a_1, \ldots, a_{h-1}, a_{h-1}, \ldots, a_1, 2a_0}] & \text{if } k = 2h-1, \\ [a_0, \overline{a_1, \ldots, a_{h-1}, a_h, a_{h-1}, \ldots, a_1, 2a_0}] & \text{if } k = 2h. \end{cases}$$

The smallest solution of $x^2 - Dy^2 = \pm 1$ is given by

$$\eta = \begin{cases} A_{2h-2} + B_{2h-2}\sqrt{D} & \text{if } k = 2h-1, \\ A_{2h-1} + B_{2h-1}\sqrt{D} & \text{if } k = 2h. \end{cases}$$

Post and Zassenhaus point out that

$$\begin{array}{l} A_{2h-2} = A_{h-1}B_{h-1} + A_{h-2}B_{h-2} \\ B_{2h-2} = (B_{h-1}^2 + B_{h-2}^2) \end{array} \quad \text{if } k = 2h-1,$$

$$\begin{array}{l} A_{2h-1} = B_{h-1}(A_h + A_{h-2}) + (-1)^h \\ B_{2h-1} = B_{h-1}(B_h + B_{h-2}) \end{array} \quad \text{if } k = 2h.$$

Also if $(P_n + \sqrt{D})/Q_n$ denotes the n-th complete quotient of the continued fraction expansion of $\sqrt{D}$, then the equations

$$\begin{array}{ll} Q_h = Q_{h-1} & \text{if } k = 2h-1, \\ P_h = P_{h+1} & \text{if } k = 2h, \end{array}$$

enable us to detect the end of the half period.

For example:

(a) $D = 21$. Here $k = 6,\ h = 3,\ P_3 = P_4$ and $\eta = 55 + 12\sqrt{21}$:

i	0	1	2	3	4	5	6
P_i	4	4	1	3	3	1	4
Q_i	1	5	4	3	4	5	1
a_i	4	1	1	2	1	1	16
A_i/B_i	4/1	5/1	9/2	23/5	32/7	55/12	912/199

(b) $D = 29$. Here $k = 5,\ h = 3,\ Q_2 = Q_3$ and $\eta = 70 + 13\sqrt{21}$:

i	0	1	2	3	4	5
P_i	5	5	3	2	3	5
Q_i	1	4	5	5	4	1
a_i	5	2	1	1	2	10
A_i/B_i	5/1	11/2	16/3	27/5	70/13	727/135

We generalise these equations, adding two others, as follows:

(i) Let $k = 2h - 1$. Then for $0 \le t \le h - 1$, we have

$$\begin{aligned} A_{2h-2} &= A_{h+t-1}B_{h-t-1} + A_{h+t-2}B_{h-t-2} & (1)\\ B_{2h-2} &= B_{h+t-1}B_{h-t-1} + B_{h+t-2}B_{h-t-2} & (2)\\ DB_{2h-2} &= A_{h+t-1}A_{h-t-1} + A_{h+t-2}A_{h-t-2} & (3)\\ A_{2h-2} &= B_{h+t-1}A_{h-t-1} + B_{h+t-2}A_{h-t-2}. & (4) \end{aligned}$$

From equations (1) and (2) give

$$A_{h+t-2} = (-1)^{h+t}(-A_{2h-2}A_{h-t-1} + DB_{2h-2}B_{h-t-1}) \qquad (5)$$

and equations (3) and (4) give

$$B_{h+t-2} = (-1)^{h+t}(A_{2h-2}B_{h-t-1} - B_{2h-2}A_{h-t-1}). \qquad (6)$$

We can combining equations (5) and (6) to get

$$A_{h+t-2} + B_{h+t-2}\sqrt{D} = (-1)^{h+t+1}\eta(-A_{h-t-1} + B_{h-t-1}\sqrt{D}). \qquad (7)$$

Consequently an equation of the form $A_{h-t-1}^2 - DB_{h-t-1}^2 = N$ gives rise to $A_{h+t-2}^2 - DB_{h+t-2}^2 = -N$.

(ii) Let $k = 2h$. Then for $0 \le t \le h-1$, we have

$$\begin{aligned} A_{2h-1} &= A_{h+t-1}B_{h-t} + A_{h+t-2}B_{h-t-1} & (8)\\ B_{2h-1} &= B_{h+t-1}B_{h-t} + B_{h+t-2}B_{h-t-1} & (9)\\ DB_{2h-1} &= A_{h+t-1}A_{h-t} + A_{h+t-2}A_{h-t-1} & (10)\\ A_{2h-1} &= B_{h+t-1}A_{h-t} + B_{h+t-2}A_{h-t-1}. & (11) \end{aligned}$$

From equations (8) and (10) give

$$A_{h+t-1} = (-1)^{h+t}(A_{2h-1}A_{h-t-1} - DB_{2h-1}B_{h-t-1}) \qquad (12)$$

and equations (9) and (11) give

$$B_{h+t-1} = (-1)^{h+t}(-A_{2h-1}B_{h-t-1} + B_{2h-1}A_{h-t-1}). \qquad (13)$$

We can combine equations (12) and (13) to get

$$A_{h+t-1} + B_{h+t-1}\sqrt{D} = (-1)^{h+t+1}\eta(-A_{h-t-1} + B_{h-t-1}\sqrt{D}). \qquad (14)$$

Consequently an equation of the form $A_{h-t-1}^2 - DB_{h-t-1}^2 = N$ gives rise to $A_{h+t-1}^2 - DB_{h+t-1}^2 = N$.

Lemma. Let $\sqrt{D} = [a_0, \overline{a_1, \ldots, a_{k-1}, 2a_0}]$. Then

$$\begin{aligned} DB_{k-1} &= a_0A_{k-1} + A_{k-2} & (15)\\ A_{k-1} &= a_0B_{k-1} + B_{k-2}. & (16) \end{aligned}$$

Proof.

$$\begin{aligned} \sqrt{D} &= [a_0, \ldots, a_{k-1}, 2a_0 + (\sqrt{D} - a_0)]\\ &= [a_0, \ldots, a_{k-1}, a_0 + \sqrt{D}]\\ &= \frac{A_{k-1}(a_0 + \sqrt{D}) + A_{k-2}}{B_{k-1}(a_0 + \sqrt{D}) + B_{k-2}}. \end{aligned}$$

The desired result then follows by cross-multiplying and equating corresponding coefficients.

We now derive equations (1)–(4). We start from the matrix identity

$$\begin{bmatrix} a_0 & 1\\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{2h-2} & 1\\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_{2h-2} & A_{2h-3}\\ B_{2h-2} & B_{2h-3} \end{bmatrix}. \qquad (17)$$

Now let $0 \le t \le h-2$. Then $h+t \le 2h-2$ and we can partition the above matrix product as

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{h+t-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{h+t} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{2h-2} & 1 \\ 1 & 0 \end{bmatrix}.$$

But $a_{h+i} = a_{h-i-1}$ for $i = 0, \ldots, h-2$, so (17) becomes

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{h+t-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{h-t-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_{2h-2} & A_{2h-3} \\ B_{2h-2} & B_{2h-3} \end{bmatrix}. \tag{18}$$

Multiplying both sides of (18) on the right by $\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$ then gives

$$\begin{aligned} \begin{bmatrix} A_{h+t-1} & A_{h+t-2} \\ B_{h+t-1} & B_{h+t-2} \end{bmatrix} \begin{bmatrix} A_{h+t-1} & A_{h+t-2} \\ B_{h+t-1} & B_{h+t-2} \end{bmatrix}^t &= \begin{bmatrix} a_0 A_{2h-2} + A_{2h-3} & A_{2h-2} \\ a_0 B_{2h-2} + B_{2h-3} & B_{2h-2} \end{bmatrix} \\ &= \begin{bmatrix} DB_{2h-2} & A_{2h-2} \\ A_{2h-2} & B_{2h-2} \end{bmatrix}, \end{aligned} \tag{19}$$

by the Lemma. Hence

$$\begin{bmatrix} A_{h+t-1} & A_{h+t-2} \\ B_{h+t-1} & B_{h+t-2} \end{bmatrix} \begin{bmatrix} A_{h+t-1} & B_{h+t-1} \\ A_{h+t-2} & B_{h+t-2} \end{bmatrix} = \begin{bmatrix} DB_{2h-2} & A_{2h-2} \\ A_{2h-2} & B_{2h-2} \end{bmatrix}. \tag{20}$$

Finally, equation (20) is equivalent to equations (1)–(4).
Similarly we derive equations (5)–(9). We start from the matrix identity

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{2h-1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_{2h-1} & A_{2h-2} \\ B_{2h-1} & B_{2h-2} \end{bmatrix}. \tag{21}$$

Now let $0 \le t \le h-1$. Then $h+t \le 2h-1$ and we can partition the above matrix product as

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{h+t-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{h+t} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{2h-1} & 1 \\ 1 & 0 \end{bmatrix}.$$

But $a_{h+i} = a_{h-i}$ for $i = 0, \ldots, h-1$, so (21) becomes

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{h+t-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{h-t} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_{2h-1} & A_{2h-2} \\ B_{2h-1} & B_{2h-2} \end{bmatrix}. \tag{22}$$

Multiplying both sides of (22) on the right by $\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$ then gives

$$\begin{aligned}\begin{bmatrix} A_{h+t-1} & A_{h+t-2} \\ B_{h+t-1} & B_{h+t-2} \end{bmatrix} \begin{bmatrix} A_{h-t} & A_{h-t-1} \\ B_{h-t} & B_{h-t-1} \end{bmatrix}^t &= \begin{bmatrix} a_0 A_{2h-1} + A_{2h-2} & A_{2h-1} \\ a_0 B_{2h-1} + B_{2h-2} & B_{2h-1} \end{bmatrix} \\ &= \begin{bmatrix} DB_{2h-1} & A_{2h-1} \\ A_{2h-1} & B_{2h-1} \end{bmatrix}, \qquad (23)\end{aligned}$$

again by the Lemma. Hence

$$\begin{bmatrix} A_{h+t-1} & A_{h+t-2} \\ B_{h+t-1} & B_{h+t-2} \end{bmatrix} \begin{bmatrix} A_{h+t} & B_{h+t} \\ A_{h+t-1} & B_{h+t-1} \end{bmatrix} = \begin{bmatrix} DB_{2h-1} & A_{2h-1} \\ A_{2h-1} & B_{2h-1} \end{bmatrix}. \quad (24)$$

Finally, equation (24) is equivalent to equations (5)–(9).

We now prove equation (6), page 104 from Perron's *Kettenbrüche*.
Theorem. Let $\sqrt{D} = [a_0, \overline{a_1, \ldots, a_k}]$. Then if $n \geq 1$, we have

$$\begin{aligned} A_{n+k-1} &= B_{n-1} D B_{k-1} + A_{n-1} A_{k-1} \\ B_{n+k-1} &= B_{n-1} A_{k-1} + A_{n-1} B_{k-1}. \end{aligned}$$

Equivalently

$$A_{n+k-1} + B_{n+k-1}\sqrt{D} = (A_{n-1} + B_{n-1}\sqrt{D})(A_{k-1} + B_{k-1}\sqrt{D}).$$

Proof. We start from the matrix identity

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n+k-1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_{n+k-1} & A_{n+k-2} \\ B_{n+k-1} & A_{n+k-2} \end{bmatrix}.$$

We partition the matrix product:

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{k-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence

$$\begin{aligned}\begin{bmatrix} A_{n+k-1} & A_{n+k-2} \\ B_{n+k-1} & B_{n+k-2} \end{bmatrix} &= \begin{bmatrix} A_{k-1} & A_{k-2} \\ B_{k-1} & B_{k-2} \end{bmatrix} \begin{bmatrix} 2a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -a_0 \end{bmatrix} \begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix} \\ &= \begin{bmatrix} A_{k-1} & A_{k-2} \\ B_{k-1} & B_{k-2} \end{bmatrix} \begin{bmatrix} 1 & a_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix} \\ &= \begin{bmatrix} A_{k-1} & a_0 A_{k-1} + A_{k-2} \\ B_{k-1} & a_0 B_{k-1} + B_{k-2} \end{bmatrix} \begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix}.\end{aligned}$$

Now identities (2) on page 102 of Perron's book state that

$$\begin{aligned} A_{n-1} &= B_{n-1}P_n + B_{n-2}Q_n \\ DB_{n-1} &= A_{n-1}P_n + A_{n-2}Q_n. \end{aligned}$$

Then taking $n = k$ and observing that $P_k = a_0$ and $Q_k = 1$, we get

$$\begin{aligned} A_{k-1} &= B_{k-1}a_0 + B_{k-2} \\ DB_{k-1} &= A_{k-1}a_0 + A_{k-2}. \end{aligned}$$

Hence

$$\begin{bmatrix} A_{n+k-1} & A_{n+k-2} \\ B_{n+k-1} & B_{n+k-2} \end{bmatrix} = \begin{bmatrix} A_{k-1} & DB_{k-1} \\ B_{k-1} & A_{k-1} \end{bmatrix} \begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix},$$

as required.

References

[1] M. Pohst and H. Zassenhaus, *On unit computation in real quadratic fields*, *EUROSAM '79*, Springer Lecture Notes in Computer Science, Volume 72, (1979) 140-152.

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