A survey on Flow Polynomial

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Abstract

This expository article is about nowhere-zero flows and the flow polynomial, which counts the number of nowhere-zero flows of a graph. Following the definitions and properties of the flow polynomial, some examples and calculations are used to illustrate and develop the arithmetic of the flow polynomial. Furthermore, the flow polynomial of some classes of graphs are computed.

1 Introduction

Much information about the flow polynomial can be found in [4], [7] and [8]. Given a graph \( G(V, E) \) with vertex set \( V \) and edge set \( E \), where multiple edges are allowed, let \((D, f)\) be an ordered pair where \( D \) is an orientation of \( E(G) \) and \( f: E(G) \rightarrow \mathbb{Z} \) be an integer-valued function called a flow. An oriented edge of \( G \) is called an arc. For a vertex \( v \in V(G) \), let \( E^+(v) = \{ \text{all arcs of } D(G) \text{ with their tails at } v \} \) and \( E^-(v) = \{ \text{all arcs of } D(G) \text{ with their heads at } v \} \).

**Definition 1.1** A \( \lambda \)-flow of a graph \( G \) is a flow \( f \) such that \( |f(e)| < \lambda \) for every edge \( e \in E(G) \) and for every vertex \( v \in V(G) \)

\[
\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e) \pmod{\lambda}.
\]

The **support** of \( f \), \( \text{supp}(f) \), is the set of all edges of \( G \) with \( f(e) \neq 0 \). A \( \lambda \)-flow is **nowhere-zero** if \( \text{supp}(f) = E(G) \).
Some long-standing conjectures on flows found in [9] are as follows:

**Conjecture 1.2** Every bridgeless graph admits a nowhere-zero 5-flow.

**Conjecture 1.3** Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.

**Conjecture 1.4** Every bridgeless graph containing no 3-edge-cut admits a nowhere-zero 3-flow.

A number $\Lambda(G)$ of interest is the least integer $\Lambda$ such that $G$ has a nowhere-zero $\Lambda$-flow. In [2], Jaeger increased the plausibility of conjecture 1.2 by proving that every bridgeless graph has a nowhere-zero 8-flow. Seymour [5], improved this upper bound $\Lambda$ to 6. In [6], $\Lambda(G)$ was further lowered for certain classes of graphs. For a graph $G(V, E)$, the cyclomatic number of $G$, $\nu(G)$ is defined as $\nu(G) = |E(G)| - |V(G)| + \kappa(G)$ where $\kappa(G)$ denotes the number of components. In [8], Tutte defines the flow polynomial, $F(G, \lambda)$, of a graph $G$ as a graph function and as a polynomial in an indeterminate $\lambda$ with integer coefficients by

$$F(G, \lambda) = (-1)^{|E(G)|} \sum_{S \subseteq E(G)} (-1)^{|S|} \lambda^{\nu(G:S)}$$

where $(G : S)$ denotes the spanning subgraph of $G$ with edge-set $S$. $F(G, \lambda)$ is a polynomial in $\lambda$ which gives the number of nowhere-zero $\lambda$-flows in $G$ independent of the chosen orientation. Tutte [8] defines the chromatic polynomial, $P(G, \lambda)$, of a graph $G$ by

$$P(G, \lambda) = \sum_{S \subseteq E(G)} (-1)^{|S|} \lambda^{\kappa(G:S)}.$$
Property 1.5 $F(G, \omega)$ is a polynomial of degree $\nu = \nu(G)$. Coefficient of $\omega^\nu$ is $(-1)^{\nu}$ and all terms in $F(G, \omega)$ have the same sign.

Property 1.6 If $G$ has no edges, then $F(G, \lambda) = 1$.

Property 1.7 If $G$ has a bridge, then $F(G, \lambda) = 0$.

Property 1.8 If $G$ consists of two graphs $H$ and $K$ which are either disjoint or have a single vertex in common, then $F(G, \lambda) = F(H, \lambda) \cdot F(K, \lambda)$.

Property 1.9 If $G$ is a cycle, then $F(G, \lambda) = \lambda - 1$.

Property 1.10 If $e$ is any edge of $G$, then $F(G, \lambda) = F(G''', \lambda) - F(G', \lambda)$, where $G'$ and $G'''$ are obtained from $G$ by deleting and contracting the edge $e$, respectively.

Property 1.11 $F(G, \lambda)$ is a topological invariant and hence any two homeomorphic graphs will have the same flow polynomial.

By a result of Jaeger [1], if $G$ is planar, then $P(G^*, \lambda) = \lambda \cdot F(G, \lambda)$, where $G^*$ is the planar dual of $G$.

2 Some Examples and Calculations

To illustrate the properties discussed above, we compute the flow polynomial of some graphs.

Example 2.1 Given $G$, $H$ and $K$ in Figure 2, $F(G, \lambda) = F(H, \lambda) = F(K, \lambda)$.

![Figure 2: Homeomorphic graphs & suppression of degree 2 vertices](image)

Example 2.2 Given $G$, $G^*$, $H$ and $H^*$ in Figure 3, we have

$$F(G, \lambda) = \frac{1}{\lambda} P(G^*, \lambda) = \frac{1}{\lambda} \left[ \lambda(\lambda - 1)(\lambda - 2) \right] = (\lambda - 1)(\lambda - 2)$$

$$F(H, \lambda) = \frac{1}{\lambda} P(H^*, \lambda) = \frac{1}{\lambda} \left[ \lambda(\lambda - 1)^3 \right] = (\lambda - 1)^3$$
Example 2.3 Let $X_3$ denote the 2-connected graph on 2 vertices with 3 edges. As we just saw, $F(X_3, \lambda) = \lambda^2 - 3\lambda + 2$ and the number of all nowhere-zero 4-flows of $X_3$ is $F(X_3, 4) = 6$. In Figure 4 we list all of them for the arbitrary orientation that we have picked for $X_3$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{The 6 nowhere-zero 4-flows of $X_3$}
\end{figure}

Example 2.4 Let $X_5$ denote the 2-connected graph on 2 vertices with 5 edges. By Lemma 4.1, $F(X_5, \omega) = \omega + \omega^2 + \omega^3 + \omega^4$. Hence the number of all nowhere-zero 3-flows of $X_5$ is $F(X_5, -2) = 10$. In Figure 5 we list all of them for the arbitrary orientation that we have picked for $X_5$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The 10 nowhere-zero 3-flows of $X_5$}
\end{figure}

Example 2.5 Let $K_4$ be the complete graph on 4 vertices with flows as shown in Figure 6. Since $F(K_4, \lambda) = \frac{1}{2}P(K_4^*, \lambda) = \frac{1}{4}P(K_4, \lambda) = -6 + 11\lambda - 6\lambda^2 + \lambda^3$, the number of nowhere-zero 4-flows of $K_4$ is $F(K_4, 4) = 6$. In Figure 6 we list all of them for the arbitrary orientation that we have picked for $K_4$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{The 6 nowhere-zero 4-flows of $K_4$}
\end{figure}
3 Sheaf Removal and Duality

Given a graph $M$ consider a bundle of multiplicity $n$ and let $K$ be the graph obtained by contracting this bundle in $G$ to a vertex and $H$ that obtained by deleting this bundle. By using Property 1.10 of flow polynomials repeatedly, Read and Whitehead [4] arrive at the “SRF”, or the Sheaf Removal Formula:

$$F(M, \omega) = (-1)^n \left[ \frac{\omega^n - 1}{1 - \omega} F(K, \omega) + F(H, \omega) \right]. \quad (3.1)$$

In Equation 3.2, $\mu(U)$ is the sum of the number of edges in the bundles of $U$, while in Equation 3.3, $\mu(U)$ denotes the sum of lengths of the chains of edge-set bundles.
in $U$. Equation 3.2 will be explained later. Here we show how Equation 3.3 works by letting $G = K_4$. We find all the spanning subgraphs $Y$ of $K_4$ and list the number of different ones in each class.

![Figure 8: $Y$, Spanning subgraphs $K_4$](image)

Next for each $Y$, we find its complement $U$.

![Figure 9: $U$, Spanning complements of $Y$](image)

Using Equation 3.3 and Property 1.7, we obtain

$$P(K_4, \omega) = \frac{(-1)^6}{\lambda^2} \left[ 1 \cdot (\omega^6) + 0 + 0 + 0 - 4\omega(\omega^3) + 0 + 0 + 0 
- 3\omega(\omega^2) + 6(\omega + \omega^2) \cdot (\omega^4) + (-2\omega - 3\omega^2 - \omega^3) \cdot (\omega^0) \right]$$

$$= \frac{(-1)^6}{\lambda^2} (\omega^6 - 4\omega^4 + 2\omega^3 + 3\omega^2 - 2\omega) = \omega^4 + 2\omega^3 - \omega^2 - 2\omega$$

$$P(K_4, \omega) = (1 - \omega)(-\omega)(-1 - \omega)(-2 - \omega) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$$
A graph $G$ with loops has no proper vertex coloring, i.e., $P(G, \lambda) = 0$. Likewise, we have already seen that a graph $M$ with bridges can not have any nowhere-zero $\lambda$-flows, i.e., $F(M, \lambda) = 0$.

Two graphs are homeomorphic if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges. Graphs with the same underlying simple graph were given the name amallamorphs by Read and Whitehead in [4]. Two graphs $G$ and $H$ are said to be chromatically equivalent if $P(G, \lambda) = P(H, \lambda)$, while two graphs $G$ and $H$ are said to be flow equivalent if $F(G, \lambda) = F(H, \lambda)$.

![Figure 10: Amallamorphic graphs and homeomorphic graphs](image)

Since multiple edges have no effect on the colorings of the vertices involved, all amallamorphic graphs have the same chromatic polynomial. Similarly by Property 1.11, all degree 2 vertices can be suppressed and that is why all homeomorphic graphs have the same flow polynomial. We summarize the above in the Table 11. The decomposition property for colorings holds, if $G$ consists of two graphs $H$ and $K$ which are disjoint.

<table>
<thead>
<tr>
<th>Property</th>
<th>Vertex $\lambda$-Colorings</th>
<th>Nowhere-Zero $\lambda$-Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynornal</td>
<td>chromatic $P(G)$</td>
<td>flow $F(G)$</td>
</tr>
<tr>
<td>Degree</td>
<td>$</td>
<td>V</td>
</tr>
<tr>
<td>Unity</td>
<td>no vertices $P(G) = 1$</td>
<td>no edges $F(G) = 1$</td>
</tr>
<tr>
<td>Reduction</td>
<td>$P(G) = P(G - e) - P(G_e)$</td>
<td>$F(G) = F(G_e) - F(G - e)$</td>
</tr>
<tr>
<td>Decomposition</td>
<td>$P(G) = P(H) \cdot P(K)$</td>
<td>$F(G) = F(H) \cdot F(K)$</td>
</tr>
<tr>
<td>Annihilation</td>
<td>with loops, $P(G) = 0$</td>
<td>with bridges, $F(G) = 0$</td>
</tr>
<tr>
<td>Equivalence</td>
<td>amallamorphism</td>
<td>homeomorphism</td>
</tr>
<tr>
<td>Main Result</td>
<td>4-Color Theorem</td>
<td>Nowhere-Zero 6-Flow</td>
</tr>
<tr>
<td>Connection</td>
<td>Jaeger: If $G$ is planar,</td>
<td>then $P(G) = \lambda F(G^*)$</td>
</tr>
</tbody>
</table>

Table 11: The comparison of colorings and flows

### 4 The Fundamental Elements

Let $X_n$ denote the 2-connected graph on 2 vertices with $n$ edges and $L_n$ denote the graph with $n$ loops and one vertex. One might call these the fundamental elements from which other flow polynomials are computed.
Figure 12: The graphs $X_n$ and $L_n$

Lemma 4.1 \( F(L_n, \omega) = (-\omega)^n \), \( F(X_n, \omega) = (-1)^{n+1} \sum_{i=1}^{n-1} \omega^i \) for \( n \geq 2 \).

Proof: By Property 1.9 we know that the flow polynomial of the cycle is \(-\omega\). By Property 1.8 \( F(L_n, \omega) = (-\omega)(-\omega) \ldots (-\omega) = (-\omega)^n \). As for $X_n$, use induction on the number of edges in $X_n$. For $n = 2$, \( F(X_2, \omega) = -\omega \).

Suppose \( F(X_n, \omega) = (-1)^{n+1} \sum_{i=1}^{n-1} \omega^i \). Take $X_{n+1}$ and apply Property 1.10 to any edge $e$. Then

\[
F(X_{n+1}, \omega) = -F(X_n) + F(L_n) = -(-1)^{n+1} \sum_{i=1}^{n-1} \omega^i + (-\omega)^n
\]

\[
= (-1)^{n+2} \sum_{i=1}^{n-1} \omega^i + (-1)^{n+2} (\omega)^n = (-1)^{n+2} \sum_{i=1}^{n} \omega^i \]

\[
\]

Figure 13: Applying the Deletion-Contraction Principle to $X_{n+1}$

5 Graphs With Prescribed Multiplicities

Figure 14: $M_3(a, b, 1), M_4(a, b, c, 1), M_n(a_1, \ldots, a_{n-1}, 1)$ and $M_n(a_1, \ldots, a_n)$
We now focus our attention on $M_n(a_1, a_2, \ldots, a_n)$, whose underlying simple graphs are the circuits, $C_n$.

**Theorem 5.1** Let $C_n$ be the underlying simple graph of the graph $M_n$ with edge multiplicities $a_1, a_2, \ldots, a_{n-1}, 1$. Then

$$F(M_n, \omega) = (-1)^{a_1 + a_2 + \cdots + a_{n-1} + 2 - n} \omega \prod_{j=1}^{n-1} a_j^{-1} \sum_{i=0}^{j} \omega^i.$$ 

**Figure 15:** Applying SRF to $M_n$

**Proof:** We use induction on $n$ and apply the SRF to the bundle whose edge multiplicity is $a_{n-1}$. Contraction and deletion of this edge bundle yields the graphs $C$ and $D$ shown in Figure 15.

$$F(M_n, \omega) = (-1)^{a_{n-1}} \left[ \frac{\omega^{a_{n-1}} - 1}{1 - \omega} F(C) + F(D) \right]$$

$$= (-1)^{a_{n-1}} \left[ - (1 + \omega + \cdots + \omega^{a_{n-1} - 1})F(C) + 0 \right] = (-1)^{a_{n-1} - 1}$$

$$(1 + \omega + \cdots + \omega^{a_{n-1} - 1})(-1)^{a_1 + a_2 + \cdots + a_{n-2} + 2 - n + 1} \omega \prod_{j=1}^{n-2} a_j^{-1} \sum_{i=0}^{j} \omega^i$$

$$= (-1)^{a_1 + a_2 + \cdots + a_{n-1} + 2 - n} \omega \prod_{j=1}^{n-1} a_j^{-1} \sum_{i=0}^{j} \omega^i$$

In the above, $D$ had a bridge. Therefore, by Property 1.7 $F(D) = 0$. □

**Figure 16:** $M_n$, $M_{n-1}$ and $P_n$ with all edge multiplicities 2
**Theorem 5.2** Let $C_n$ be the underlying simple graph of the graph $M_n$ with all edge multiplicities 2. For $n \geq 2$, $F(M_n, \omega) = (-1)^{n+1}[\omega(1 + \omega)^n - \omega^n]$.

**Proof:** We proceed by induction. For $n = 2$, $F(M_2, \omega) = F(X_4, \omega) = -\omega - \omega^2 - \omega^3 = (-1)^{2+1}[\omega(1 + \omega)^2 - \omega^2]$. Suppose that $F(M_{n-1}, \omega) = (-1)^n[\omega(1 + \omega)^{n-1} - \omega^{n-1}]$. We apply the SRF to any bundle of edge multiplicity 2. Contraction and deletion of this edge bundle yields the graphs $M_{n-1}$ and $P_n$ shown in Figure 16. Hence we have

$$F(M_n, \omega) = (-1)^2 \left[ \frac{\omega^2 - 1}{1 - \omega} F(M_{n-1}, \omega) + F(P_n, \omega) \right]$$

$$= -(1 + \omega)(-1)^n \left[ \omega(1 + \omega)^{n-1} + (-1)^{n-1} \omega^{n-1} \right] + (-\omega)^{n-1}$$

$$= (-1)^{n+1} \omega(1 + \omega)^n + (-1)^n \omega^n$$

$$+ (-1)^{n-1} \omega^{n-1} = (-1)^{n+1} \omega(1 + \omega)^n + (-1)^n \omega^n \quad \Box$$

Now we try to find the flow polynomial of the general cycle graph $M_n(a_1, a_2, \ldots, a_n)$ depicted in Figure 17.

**Theorem 5.3** Assume $M_n(a_1, a_2, \ldots, a_n)$ has $C_n$ as its underlying simple graph with edge multiplicities $a_1, a_2, \ldots, a_n$. Then for $n \geq 3$

$$F(M_n, \omega) = (-1)^{\sum_{i=1}^{n-2}(a_{n+1-i})} \cdot F(M_2, \omega) \cdot \frac{n-2}{(1 - \omega)^{n-2}} \cdot \prod_{j=1}^{n-2} (\omega^{a_{n+1-j}} - 1)$$

$$+ \sum_{j=1}^{n-2} \left( (-1)^{\sum_{i=1}^{n-2}(a_{n+1-i})} \cdot F(P_{j+2}, \omega) \cdot \prod_{m=1}^{n-2-j} \left( \frac{\omega^{a_{n+1-m}} - 1}{(1 - \omega)^{n-2-j}} \right) \right)$$

![Figure 17: $M_n(a_1, \ldots, a_n)$, $M_{n-1}(a_1, \ldots, a_{n-1})$ and $P_n(a_1, \ldots, a_{n-1})$](image)

**Proof:** Here we let $M_1 = M(a_1, \ldots, a_i)$, while $P_i = P(a_1, \ldots, a_{i-1})$ and $M_2 = M_2(a_1, a_2)$ are the graphs shown in Figure 18. Also $M_2 \cong X_{a_1 + a_2}$. We proceed by induction. For $n = 3,$
Figure 18: The graphs $P_i$ and $M_i$

\[
F(M_3, \omega) = (-1)^{a_3} \frac{F(M_2, \omega)}{1 - \omega} \cdot (\omega^{a_3} - 1) + (-1)^{a_3} F(P_3, \omega).
\]

However, the above is merely an application of the deletion-contraction principle for the edge bundle $a_3$ of $M_3$. Now suppose that $F(M_k, \omega)$ is known. We apply the SRF to the bundle of edge multiplicity $a_k+1$ of $M_{k+1}$. Contraction and deletion of this edge bundle yields the graphs $M_k$ and $P_{k+1}$. Hence we have

\[
F(M_{k+1}, \omega) = (-1)^{a_{k+1}} \left\{ \frac{\omega^{a_{k+1}} - 1}{1 - \omega} F(M_k, \omega) + F(P_{k+1}, \omega) \right\}
\]

\[
= (-1)^{a_{k+1}} \left\{ \frac{\omega^{a_{k+1}} - 1}{1 - \omega} \left[ (-1)^{a_{k+1}-1} F(M_2, \omega) \cdot \prod_{j=1}^{k-2} (\omega^{a_{k+1}-j} - 1) \right] \right. \\
+ \sum_{j=1}^{k-2} \left( (-1)^{a_{k+1}-j} F(P_{j+2}, \omega) \cdot \prod_{m=1}^{k-2-j} \left( \frac{\omega^{a_{k+1}-m} - 1}{1 - \omega} \right) \right) \\
+ \left. F(P_{k+1}, \omega) \right\}
\]

\[
= (-1)^{a_{k+1}} \left\{ \frac{\omega^{a_{k+1}} - 1}{1 - \omega} \left[ (-1)^{a_{k+1}-1} F(M_2, \omega) \cdot \prod_{j=0}^{k-2} (\omega^{a_{k+1}-j} - 1) \right] \right. \\
+ \sum_{j=1}^{k-1} \left( (-1)^{a_{k+1}-j} F(P_{j+2}, \omega) \cdot \prod_{m=1}^{k-1-j} \left( \frac{\omega^{a_{k+1}-m} - 1}{1 - \omega} \right) \right) \\
+ \left. F(P_{k+1}, \omega) \right\}
\]

However, at this point a simple shift in all the indices will change the last statement to the following.

\[
F(M_{k+1}, \omega) = (-1)^{a_{k+2}} \cdot \frac{F(M_2, \omega)}{1 - \omega} \cdot \prod_{j=1}^{k-1} (\omega^{a_{k+1}-j} - 1) \\
+ \sum_{j=1}^{k-1} \left( (-1)^{a_{k+2}-j} F(P_{j+2}, \omega) \cdot \prod_{m=1}^{k-1-j} \left( \frac{\omega^{a_{k+2}-m} - 1}{1 - \omega} \right) \right)
\]

And this is exactly what we wanted to show. □
We now express the flow polynomial of $M_3$ not as a rational function. Table 20 provided some insight as to what the formula should be.

![Figure 19: The graph $M_3(a, b, c)$](image)

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$\nu$</th>
<th>coefficients of $F(M_3, \omega)$ in ascending powers of $\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>8</td>
<td>11</td>
<td>-1, -3, -4, -4, -4, -3, -3, -2, -1</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>16</td>
<td></td>
<td>1, 3, 5, 7, 9, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>17</td>
<td></td>
<td>-1, -3, -5, -7, -8, -9, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>18</td>
<td></td>
<td>1, 3, 5, 7, 8, 9, 10, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1</td>
</tr>
</tbody>
</table>

Table 20: Flow polynomial of $M_3(a, b, c)$ for selected edge multiplicities

Notice that the coefficients, in absolute value, start at 1 and increase through the odd numbers, then go up by consecutive integers, reach a plateau and stay there for a while and then decrease back to 1.

**Theorem 5.4** Let $K_3$ be the underlying simple graph of the graph $M_3$ with edge multiplicities $a, b, c$ where $0 < a \leq b \leq c$, and $a, b, c \in \mathbb{N}$. Then

$$F(M_3, \omega) = (-1)^{a+b+c} \left[ \sum_{i=1}^{a} (2i-1)\omega^i + \sum_{i=a+1}^{b} (a + i)\omega^i ight] + \sum_{i=b+1}^{a+b+c-2} (a + b - 1)\omega^i + \sum_{i=c+1}^{a+b+c} (a + b + c - 1 - i)\omega^i \right].$$

![Figure 21: SRF applied to the bundle $a$](image)

**Proof:** We apply the SRF to the bundle whose edge multiplicity is $a$. Contraction and deletion of edge bundle $a$ is depicted in Figure 21.

$$F(M_3, \omega) = (-1)^a \left[ \frac{\omega^a - 1}{1 - \omega} F(X_{b+c}, \omega) + F(X_b, \omega) \cdot F(X_c, \omega) \right]$$

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Upon multiplying out and collecting the terms in the products $P$ and $Q$, we see a number of breaks in the ascending powers of $\omega$ where these powers can be linearly ordered. We gather the similar terms and add the coefficients of $\omega^i$ in Table 22. Upon adding all the terms, the result follows.

**Table 22: Collection of coefficients of $\omega^i$**

<table>
<thead>
<tr>
<th>Power of $\omega$</th>
<th>1 2 a b c b+c-2 b+c-1 a+b+c-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power of $P$</td>
<td>1 2 a b c b+c-2 b+c-1 a+b+c-2</td>
</tr>
<tr>
<td>Power of $Q$</td>
<td>0 1 a b c b+1-1 a+1 0 0</td>
</tr>
<tr>
<td>Sum of coeff</td>
<td>1 3 a b c b+1-1 a+1 a b+c-1</td>
</tr>
</tbody>
</table>

**Figure 23:** The sector graph

In [7], it was shown that the flow polynomial of the sector graph $S_k = S_k(a_1, a_2, \ldots, a_n)$, shown in Figure 23, is

$$F(S_k, \omega) = (-1)^k \left[ \frac{\omega - 1}{1 - \omega} (1 - \omega^{a_1} - 1) \left( \prod_{i=2}^{k-1} \frac{\omega^{b_i} - 1}{1 - \omega^{b_i}} \right) \right].$$

**Theorem 5.5** Given $k \geq 2$ and $W_k(a_1, a_2, \ldots, a_k)$ whose underlying simple graph is $W_k$, the wheel with $k$ spokes, we have

$$F(W_k, \omega) = (-1)^{k+1} \sum_{i=1}^{k} a_i \cdot \omega \cdot \left[ \sum_{i=1}^{k-1} (-1)^{i+1} \right].$$
\[
\frac{(\omega^{a_{k+1}} - 1) \prod_{j=1}^{k} (\omega^{1+a_{k+1}} - 1) - 1}{(1 - \omega)^{k+1}} + (-1)^{k+1} \frac{\omega^{a_1} - \omega}{1 - \omega}.
\]

Figure 24: The wheel \(W_k(a_1, a_2, \ldots, a_k)\)

**Proof:** We proceed by induction. For \(k = 2\), the formula gives

\[
F(W_2, \omega) = (-1)^{2+a_1+a_2} \cdot \omega \left[ (-1)^2 (\omega^2 - 1)(\omega^{1+a_1} - 1) \right] \frac{\omega^{a_1} - \omega}{1 - \omega}
\]

Figure 25: The wheel \(W_2\) with a redrawing of it

To verify this, we start with \(W_2\) and apply SRF to some edge bundle, say \(a_2\). Then we obtain \(F(W_2, \omega) = \)

\[
= (-1)^{a_2} \frac{\omega^{a_2} - 1}{1 - \omega} F(X_{2+a_1}, \omega) + F(X_{a_1}, \omega) F(X_2, \omega)
\]

\[
= (-1)^{a_2} \frac{\omega^{a_2} - 1}{1 - \omega} (-1)^{1+a_1} \frac{\omega^{2+a_1} - \omega}{1 - \omega} + (-1)^{1+a_1} \frac{\omega^{a_1} - \omega}{1 - \omega} \cdot (-\omega)
\]

\[
= (-1)^{1+a_1+a_2} \frac{\omega^{a_2} - 1}{1 - \omega} \cdot \frac{-\omega(\omega^{1+a_1} - 1)}{1 - \omega} + (-1)^{1+a_1+a_2} \frac{\omega^{a_1} - \omega}{1 - \omega}
\]

\[
= (-1)^{2+a_1+a_2} \cdot \omega \left[ (-1)^2 (\omega^2 - 1)(\omega^{1+a_1} - 1) \right] \frac{\omega^{a_1} - \omega}{1 - \omega}
\]

Now suppose the result is true for \(k = n\). Start with \(W_{n+1}\) and apply SRF to the bundle whose edge multiplicity is \(a_{n+1}\). The result is shown in Figure 26.
In Figure 26, the graph obtained from $W_{n+1}$ by deletion of bundle $a_{n+1}$ is homeomorphic to $H$ by Property 1.11 where $H \cong W_n$, while the one obtained from contracting $a_{n+1}$ is $S_n$, a sector graph whose flow polynomial is known. Hence we now have

$$F(W_{n+1}, \omega) = (-1)^{a_{n+1}} \left[ \frac{\omega^{a_{n+1}} - 1}{1 - \omega} F(G, \omega) + F(H, \omega) \right]$$

$$= (-1)^{a_{n+1}} \sum_{i=1}^{n+1} a_i \cdot \omega^{a_{n+1}} - 1 \cdot \prod_{i=1}^{n+1} \left( \frac{\omega^{a_{n+1}} - 1}{1 - \omega} \right)$$

$$+ (-1)^{a_{n+1}} F(W_n, \omega) = (-1)^{n+1} \sum_{i=1}^{n+1} a_i \cdot \omega^{a_{n+1}} - 1 \cdot \prod_{i=1}^{n+1} \left( \frac{\omega^{a_{n+1}} - 1}{1 - \omega} \right)$$

$$\prod_{i=1}^{n+1} \left( \frac{\omega^{a_{n+1}} - 1}{1 - \omega} \right) + (-1)^{a_{n+1}} \left( (-1)^{n+1} \sum_{i=1}^{n+1} a_i \cdot \omega^{a_{n+1}} - 1 \cdot \prod_{i=1}^{n+1} \left( \frac{\omega^{a_{n+1}} - 1}{1 - \omega} \right) \right)$$

$$= (-1)^{n+1} \sum_{i=1}^{n+1} a_i \cdot \omega \cdot \left( \frac{\omega^{a_{n+1}} - 1}{1 - \omega} \right) \prod_{i=1}^{n+1} \left( \frac{\omega^{a_{n+1}} - 1}{1 - \omega} \right) + \sum_{i=1}^{n+1} \left( (-1)^{n+1} \omega^{a_{n+1}} - \omega \right)$$

Figure 26: Decomposition results in sector and wheel graphs
By closely studying \( \{ \ldots + [ \ldots ] \} \) in the last Equation, we can see that the first term can be absorbed by the second by lowering the index in the sum from 1 to 0.

\[
(-1)^{n+1} \sum_{i=0}^{n-1} (-1)^i \left\{ \frac{\omega^{a_{n+1-i}} - 1}{1 - \omega} - 1 \right\} + (-1)^{n+1} \omega^{a_1} - \omega \left\{ \frac{1}{1 - \omega} \right\}
\]

However now by a readjustment of the index of summation, we obtain

\[
(-1)^{n+1} \sum_{i=0}^{n} (-1)^i \left\{ \frac{\omega^{a_{n+1-i}} - 1}{1 - \omega} - 1 \right\} + (-1)^{n+1} \omega^{a_1} - \omega \left\{ \frac{1}{1 - \omega} \right\}
\]

which is the desired result and completes the inductive proof.

Figure 27: The wheel \( W_1 \)

As the reader might have noticed by now, the above induction has an initial starting point at \( n = 2 \). The first wheel \( W_1 \) is a degenerate case and must be dealt with separately. This is however a very trivial case and as Figure 27 shows, \( W_1 \) can be factored as the disjoint union of the \( X_{a_1} \) and \( L_1 \) by Property 1.8. So
\[ F(W_1, \omega) = F(X_{a_1}, \omega) \cdot F(X_2, \omega) = (-1)^{1+a_1} \frac{\omega^{a_1}-1}{\omega-1} \cdot (-\omega) = (-1)^{a_1} \frac{\omega^{a_1}-\omega}{\omega-1} \]

6 Expansion On Certain Subgraphs

As we explained in Section 3, the flow polynomial of a graph \( M \) can be expressed as a polynomial in \( \omega = 1 - \lambda \), where the coefficients of \( \omega^i \) are chromatic polynomials of certain subgraphs of \( M \). See Equation 3.2.

**Lemma 6.1** Let \( C_3 \) be the underlying simple graph of the graph \( M_3 \) with edge multiplicities \( a, b, c \). Then

\[ F(M_3, \omega) = \frac{(-1)^{a+b+c}}{(1-\omega)^3} \left[ (\omega - \omega^3) + (\omega^2 - \omega)(\omega^a + \omega^b + \omega^c) + (1 - \omega)\omega^{a+b+c} \right]. \]

**Lemma 6.2** Let \( C_4 \) be the underlying simple graph of the graph \( M_4 \) with edge multiplicities \( a, b, c, d \). Then \( F(M_4, \omega) = \)

\[ \frac{(-1)^{a+b+c+d}}{(1-\omega)^4} \left[ (\omega^4 - \omega) + (\omega^3 - \omega^2)(\omega^a + \omega^b + \omega^c + \omega^d) + (\omega^2 - \omega) \cdot \right. \]

\[ (\omega^{a+c} + \omega^{b+d} + \omega^{a+b} + \omega^{a+d} + \omega^{b+c} + \omega^{c+d}) + (1 - \omega)\omega^{a+b+c+d} \]

**Proof:** We look at all different classes of subgraphs of \( M_3 \) here and use 3.2. In the \( < U > \) column of Table 28, we list representative subgraphs. In the \( G_U \) column, the complement of each representative subgraph, with all the edges in \( U \) contracted to point, are listed.
The desired result follows.

We offer 2 different ways of obtaining a formula for the flow polynomial of the general cycle graph $M_n$.

Our first method is recursion: SRF applied to $M_n(a_1, a_2, \ldots, a_n)$ results in $M_{n-1}(a_1, a_2, \ldots, a_{n-1})$ and a $P_n(a_1, a_2, \ldots, a_{n-1})$, both of which can be assumed to have previously computed flow polynomials. In this manner, after applying SRF, we arrive at a formula for the flow polynomial of $M_n$ which is in terms of $M_{n-1}$ and $X_i$ for $i \leq n - 1$. Based on this argument, we state the following theorem:

**Theorem 6.3** Let $C_n$, the cycle of length $n$, be the underlying simple graph of the graph $M_n$ whose edge multiplicities are $a_1, a_2, \ldots, a_n$. Then

$$F(M_n, \omega) = (-1)^{a_n} \left[ \frac{\omega^{a_n} - 1}{1 - \omega} F(M_{n-1}, \omega) + F(P_n, \omega) \right]$$

$$= (-1)^{a_n} \left[ \frac{\omega^{a_n} - 1}{1 - \omega} F(M_{n-1}, \omega) + \prod_{i=1}^{n-1} F(X_{a_i}, \omega) \right].$$

The second way is to get more inspiration from Lemmas 6.1 and 6.2 by studying the columns for the powers of $\omega$ in detail. To make this task easier we make the following definition:

<table>
<thead>
<tr>
<th>$\omega^0$</th>
<th>$\omega^1$</th>
<th>$\omega^2$</th>
<th>$\omega^3$</th>
<th>$\omega^4$</th>
</tr>
</thead>
<tbody>
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<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega^3$</td>
<td>$\omega^4$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega^3$</td>
<td>$\omega^4$</td>
<td>$\omega^5$</td>
</tr>
<tr>
<td>$\omega^2$</td>
<td>$\omega^3$</td>
<td>$\omega^4$</td>
<td>$\omega^5$</td>
<td>$\omega^6$</td>
</tr>
<tr>
<td>$\omega^3$</td>
<td>$\omega^4$</td>
<td>$\omega^5$</td>
<td>$\omega^6$</td>
<td>$\omega^7$</td>
</tr>
<tr>
<td>$\omega^4$</td>
<td>$\omega^5$</td>
<td>$\omega^6$</td>
<td>$\omega^7$</td>
<td>$\omega^8$</td>
</tr>
</tbody>
</table>

**Table 28:** Subgraph expansion of $M_4$
Let the cycle of length \( n \) be the underlying simple graph of the graph \( M_n = M_n(a_1, a_2, \ldots, a_n) \) whose edge multiplicities are \( a_1, a_2, \ldots, a_n \). Then

\[
F(M_n, \omega) = \frac{(-1)^n}{(1-\omega)^n} \left[ \sum_{i=1}^{n-1} (-1)^{i+1}(\omega - \omega^{n+1-i}) \Psi(n, j \omega - 1, \omega) \right] + (1-\omega)\omega \sum_{i=1}^{n} a_i
\]  

(6.4)

Example 6.5 Let us find the flow polynomial of \( M_6 \). Using Equation 6.4, first we determine all of the \( \binom{6}{0} = 1 \) 0-subsets, \( \binom{6}{1} = 6 \) 1-subsets, \( \binom{6}{2} = 15 \) 2-subsets, \( \binom{6}{3} = 20 \) 3-subsets, \( \binom{6}{4} = 15 \) 4-subsets. Next we find \( \Psi(6, j \omega - 1, \omega) \) for \( j = 1, 2, 3, 4 \).

\[
\begin{align*}
\Psi(6, 0, \omega) &= \omega^0 = 1 \\
\Psi(6, 1, \omega) &= \omega^a + \omega^b + \omega^c + \omega^d + \omega^e + \omega^f \\
\Psi(6, 2, \omega) &= \omega^{a+b} + \omega^{b+c} + \omega^{c+d} + \omega^{d+e} + \omega^{e+f} + \omega^{f+a} + \omega^{a+c} + \omega^{b+d} \\
&\quad + \omega^{c+e} + \omega^{d+f} + \omega^{e+a} + \omega^{f+b} + \omega^{a+d} + \omega^{b+e} + \omega^{c+f} \\
\Psi(6, 3, \omega) &= \omega^{a+b+c} + \omega^{b+c+d} + \omega^{c+d+e} + \omega^{d+e+f} + \omega^{e+a} + \omega^{f+b} \\
&\quad + \omega^{a+b+d} + \omega^{b+c+e} + \omega^{c+d+f} + \omega^{d+e+a} + \omega^{e+f+b} \\
&\quad + \omega^{a+b+c+e} + \omega^{b+c+d+e} + \omega^{c+d+e+f} + \omega^{d+e+f+a} \\
&\quad + \omega^{d+e+f+a} \\
\Psi(6, 4, \omega) &= \omega^{a+b+c+d+e} + \omega^{b+c+d+e+1} + \omega^{c+d+e+f} + \omega^{d+e+f+1} + \omega^{e+f+1} \\
&\quad + \omega^{f+1} + \omega^{a+b+c} + \omega^{a+b+c+e} + \omega^{b+c+d+e+1} + \omega^{c+d+e+1} \\
&\quad + \omega^{e+f+1} + \omega^{f+1} + \omega^{a+b+d} + \omega^{a+b+c+e} + \omega^{b+c+d+e} + \omega^{b+c+d+e+1} \\
&\quad + \omega^{c+d+e+f} + \omega^{d+e+f+1} + \omega^{e+f+1} + \omega^{f+1}
\end{align*}
\]

We can now use the above \( \Psi \) values with \( n = 6 \) in Equation 6.4 to obtain

\[
F(M_6, \omega) = \frac{(-1)^{a+b+c+d+e+f}}{(1-\omega)^6} \left[ (\omega - \omega^6)\Psi(6, 0, \omega) \right] \\
- (\omega - \omega^5)\Psi(6, 1, \omega) + (\omega - \omega^4)\Psi(6, 2, \omega) - (\omega - \omega^3)\Psi(6, 3, \omega) \\
+ (\omega - \omega^2)\Psi(6, 4, \omega) + (1-\omega)\omega^{a+b+c+d+e+f}
\]
Finally, The proof of the following two results can be found in [7].

**Theorem 6.6** Let $W_n$, the wheel on $n+1$ vertices, be the underlying simple graph of the graph $G$, where the rim edges of $G$ have multiplicity 1 and the spokes of $G$ have edge multiplicities $\vec{a} = (a_1, a_2, \ldots, a_n)$. Pick any $\sigma \in S_n$ and apply $\sigma$ to the spokes of $G$ and call the new graph $G_\sigma$ whose edge multiplicities now are $\sigma(\vec{a}) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$. Then the flow polynomial of $G$ is permutation invariant, i.e.,

$$F(G, \lambda) = F(G_\sigma, \lambda).$$

**Theorem 6.7** Let $C_n$ be the underlying simple graph of the graph $G$ whose edge multiplicities are $\vec{a} = (a_1, a_2, \ldots, a_n)$. Pick any $\sigma \in S_n$ and apply $\sigma$ to the edge bundles of $G$ and call the new graph $G_\sigma$, whose edge multiplicities now are $\sigma(\vec{a}) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$. Then the flow polynomial of $G$ is permutation invariant, i.e.,

$$F(G, \omega) = F(G_\sigma, \omega).$$

**References**


