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PURELY PERIODIC NEAREST SQUARE CONTINUED FRACTIONS

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Abstract

We present a test for determining whether a real quadratic irrational has a purely periodic nearest square continued fraction expansion. This test is somewhat more explicit than the standard test and simplifies the programming of the algorithm.

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1. Introduction

Simple tests have long been known for determining whether a real quadratic irrational $\xi = (P+\sqrt{D})/Q, D > 0$ and not a perfect square, has a purely periodic regular continued fraction expansion, or nearest integer continued fraction expansion. Thus if $\overline{\xi} = (P-\sqrt{D})/Q$, then ξ has a purely periodic regular continued fraction expansion if and only if $\xi > 1$ and $-1 < \overline{\xi} < 0$ (see, e.g., [P, pp. 73–74]). Also ξ has a purely periodic nearest integer continued fraction expansion if and only if $\xi > 2$ and $(1-\sqrt{5})/2 < \overline{\xi} \leq (3-\sqrt{5})/2$ [MR].

No test for pure periodicity as simple as these is known for the *nearest square* continued fraction, defined in the next section. Instead, A. A. K. Ayyangar [A2, p. 27] gave a definition of *reduced* quadratic irrational and showed that ξ has a purely periodic nearest square continued fraction expansion if and only if ξ is reduced. In this paper, we give a more explicit version of Ayyangar's definition which is useful in detecting the start of a period.

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2. The nearest square continued fraction algorithm

This continued fraction was introduced by A. A. K. Ayyangar in 1938 and 1941 [A1, A2]. Let $\xi_0 = \frac{P+\sqrt{D}}{Q}$ be a surd in standard form, i.e.,

(i) P, Q, and D are integers, $Q \neq 0$, D is not a perfect square,

(ii) $(P^2 - D)/Q$ is an integer,

(iii) $\gcd(P, Q, (D - P^2)/Q) = 1$.

Then with $c = \lfloor \xi_0 \rfloor$, the integer part of ξ_0, we can represent ξ_0 in two ways (the *positive* and *negative* representations of ξ_0):

$$\xi_0 = c + \frac{Q'}{P' + \sqrt{D}} = c + 1 - \frac{Q''}{P'' + \sqrt{D}},$$

where $\frac{P'+\sqrt{D}}{Q'} > 1$ and $\frac{P''+\sqrt{D}}{Q''} > 1$ are also in standard form. We choose the partial denominator a_0 and numerator ϵ_1 of the new continued fraction expansion as follows:

(a) $a_0 = c$ and $\epsilon_1 = 1$, if $|Q'| < |Q''|$, or $|Q'| = |Q''|$ and $Q < 0$,

(b) $a_0 = c + 1$ and $\epsilon_1 = -1$, if $|Q'| > |Q''|$, or $|Q'| = |Q''|$ and $Q > 0$.

The term *nearest square* arises on noting that $P'^2 = D - QQ'$ and $P''^2 = D + QQ''$ and restating (a) and (b) using the following equivalence:

$$|Q'| \geq |Q''| \iff |QQ'| \geq |QQ''| \iff |P'^2 - D| \geq |P''^2 - D|.$$

Then $\xi_0 = a_0 + \frac{\epsilon_1}{\xi_1}$, where $|\epsilon_1| = 1, a_0$ is an integer and $\xi_1 = \frac{P_1+\sqrt{D}}{Q_1} > 1$. Also $P_1 = P'$ or P'' and $Q_1 = Q'$ or Q'', according as $\epsilon_1 = 1$ or -1. We call ξ_1 the *successor* of ξ_0. We proceed similarly with ξ_1, and so on. Then the complete quotients ξ_n satisfy

$$\xi_n = a_n + \frac{\epsilon_{n+1}}{\xi_{n+1}} \text{ and } \xi_0 = a_0 + \frac{\epsilon_1|}{|a_1} + \frac{\epsilon_2|}{|a_2} + \cdots + \frac{\epsilon_n|}{|\xi_n}, \tag{2.1}$$

with partial numerator $\epsilon_{i+1} = \pm 1$ and partial denominator $a_i \geq 1$ if $i \geq 1$. This expansion is called the *nearest square* continued fraction (NSCF) expansion.

Analogous relations to those for regular continued fractions also hold for P_n, Q_n and a_n:

$$P_{n+1} + P_n = a_n Q_n, \tag{2.2}$$

$$P_{n+1}^2 + \epsilon_{n+1} Q_n Q_{n+1} = D. \tag{2.3}$$

Ayyangar [A2, Theorem II, p. 25] proved that the NSCF expansion is eventually periodic, i.e., the complete quotients ξ_n eventually satisfy $\xi_i = \xi_{i+k}$ for $i \geq i_0$ for some $k \geq 1$. Then $\epsilon_{i+1} = \epsilon_{i+k+1}$ and $a_i = a_{i+k}$ for all $i \geq i_0$.

Our main result is:

Theorem 2.1. *Let $\xi = (P + \sqrt{D})/Q$ be in standard form and let $R = (D - P^2)/Q$. Then ξ has a purely periodic nearest square continued fraction expansion if and only if*

(i) $Q^2 + \frac{1}{4}R^2 \leq D, \quad \frac{1}{4}Q^2 + R^2 \leq D,$

(ii) *ξ is the successor of $1/\xi$,*

(iii) *ξ is not of the form $\frac{p+q+\sqrt{p^2+q^2}}{2q}, p > 2q > 0$.*

3. Reduced NSCF quadratic surds

Ayyangar [A2, p. 27] gave the following definition of reduced quadratic surd. He first defined a *special* surd ξ_v by the inequalities

$$Q_{v+1}^2 + \tfrac{1}{4}Q_v^2 \leq D, \quad Q_v^2 + \tfrac{1}{4}Q_{v+1}^2 \leq D, \tag{3.4}$$

then defined a *semi-reduced* surd to be the successor of a special surd. Finally a *reduced* surd is defined to be the successor of a semi-reduced surd. Ayyangar [A2, p. 28] proved that a semi–reduced surd is a special surd. Consequently a reduced surd is also semi–reduced. That a quadratic surd has a purely periodic NSCF expansion if and only if it is reduced, is proved in [A2, pp. 101–102]. To use the Ayyangar characterization to decide if a surd ξ is reduced, one has to determine if there is a special surd whose second successor is ξ; doing this can be combersome.

One example of a reduced surd is the successor of $\sqrt{D}/Q > 1$, where Q divides D (see [A2, Theorem XII, p. 102]). Another example that figures prominently in [A2] is $\frac{p+q+\sqrt{p^2+q^2}}{p}$, where $p > 2q > 0$.

Lemma 3.1. *If two different semi–reduced surds have the same successor, they have the form $\frac{p\pm q+\sqrt{D}}{2q}$, where $p > 2q > 0$.*

Proof. This is [A2, Theorem IX, p. 99]. □

Lemma 3.2. *ξ is semi–reduced if and only if ξ is reduced, or $\xi = \frac{p+q+\sqrt{p^2+q^2}}{2q}, p > 2q > 0$.*

Proof. (a) Suppose ξ is semi–reduced and let η be its successor. Then η is reduced and has a unique reduced predecessor χ, by [A2, Corollary 1, p. 101]. By Lemma 3.1, either $\xi = \chi$, or ξ and χ are equal to $\frac{p\pm q+\sqrt{D}}{2q}$, where $p > 2q > 0$. However by [A2, Theorem X, p. 100], $\frac{p+q+\sqrt{p^2+q^2}}{2q}$ has no semi–reduced predecessor and hence is not reduced, so $\chi = \frac{p-q+\sqrt{D}}{2q}$ and hence $\xi = \frac{p+q+\sqrt{p^2+q^2}}{2q}$.

(b) If ξ is reduced, it is the successor of a semi–reduced surd χ and as previously observed, this is special. Hence ξ is semi–reduced. If $\xi = \frac{p+q+\sqrt{p^2+q^2}}{2q}, p > 2q > 0$, in view of the equation

$$\frac{p - q + \sqrt{p^2 + q^2}}{p} = 1 + \frac{p}{q + \sqrt{p^2 + q^2}} = 2 - \frac{2q}{p + q + \sqrt{p^2 + q^2}},$$

as $p > 2q > 0$, we see ξ is the successor of the special surd $\frac{p-q+\sqrt{p^2+q^2}}{p}$ and is hence semi–reduced. ☐

Corollary 3.1. *ξ is reduced if and only if ξ is semi–reduced and not of the form* $\xi = \frac{p+q+\sqrt{p^2+q^2}}{2q}, p > 2q > 0$.

Ayyangar did not explicitly mention Lemma 3.2 or Corollary 3.1 in his paper [A2].

4. Some lemmas on successors

Lemma 4.3. *ξ and $-\xi$ have the same successor.*

Proof. This follows from the fact that the positive–negative represention

$$\frac{P+\sqrt{D}}{Q} = c + \frac{Q'}{P'+\sqrt{D}} = c+1-\frac{Q''}{P''+\sqrt{D}},$$

implies the positive–negative represention

$$\frac{P+\sqrt{D}}{-Q} = -c-1+\frac{Q''}{P''+\sqrt{D}} = -c-\frac{Q'}{P'+\sqrt{D}}.$$

Then the conditions defining the successor of ξ also define the same successor of $-\xi$. ☐

Lemma 4.4. *If ξ is the successor of a quadratic surd, then ξ is the successor of $1/\xi$.*

Proof. . Suppose $\xi = \frac{P+\sqrt{D}}{Q}$ is the successor of $\frac{P_0+\sqrt{D}}{Q_0}$. Then the successor equation

$$\frac{P_0+\sqrt{D}}{Q_0} = b + \epsilon\frac{Q}{P+\sqrt{D}} \tag{4.5}$$

gives $P_0 + P = bQ_0$ and $D - P^2 = \epsilon QQ_0$. We also have the positive–negative representation

$$\frac{P_0+\sqrt{D}}{Q_0} = a + \frac{Q'}{P'+\sqrt{D}} = a+1-\frac{Q''}{P''+\sqrt{D}}.$$

Then

$$\begin{aligned}\epsilon/\xi = \frac{\epsilon Q}{P+\sqrt{D}} &= \frac{\epsilon Q(\sqrt{D}-P)}{D-P^2} = \frac{\epsilon Q(\sqrt{D}-P)}{\epsilon QQ_0}\\ &= \frac{-P+\sqrt{D}}{Q_0}\\ &= \frac{P_0-bQ_0+\sqrt{D}}{Q_0}\\ &= -b+\frac{P_0+\sqrt{D}}{Q_0}\\ &= -b+a+\frac{Q'}{P'+\sqrt{D}} = -b+a+1-\frac{Q''}{P''+\sqrt{D}}\end{aligned} \tag{4.6}$$

and this positive–negative representation implies that the successor of ϵ/ξ is ξ. If $\epsilon = -1$, Lemma 4.3 implies that the successor of $1/\xi$ is also ξ. ☐

Lemma 4.5. *Let* $\xi = \frac{P+\sqrt{D}}{Q}$ *and* $R = (D-P^2)/Q$. *Then* ξ *is semi–reduced if and only if*

(i) $Q^2 + \frac{1}{4}R^2 \le D, \quad \frac{1}{4}Q^2 + R^2 \le D,$

(ii) ξ *is the successor of* $1/\xi$.

Proof. (a) Suppose $\xi = \frac{P+\sqrt{D}}{Q}$ is semi–reduced. Then ξ is the successor of a special surd $\xi_0 = \frac{P_0+\sqrt{D}}{Q_0}$. Then with ϵ as in (4.5), as before, we have $R = (D-P^2)/Q = \epsilon Q_0$ and inequalities $Q_0^2 + \frac{1}{4}Q^2 \le D$, $\frac{1}{4}Q^2 + Q_0^2 \le D$ become

$$R^2 + \tfrac{1}{4}Q^2 \le D, \quad \tfrac{1}{4}Q^2 + R^2 \le D.$$

Also by Lemma 4.4, ξ is the successor of $1/\xi$.
(b) Suppose (i) and (ii) hold. Then as $1/\xi = \frac{-P+\sqrt{D}}{R}$, (i) and (ii) imply $1/\xi$ is special and that ξ is semi–reduced. ☐

Our Theorem then follows from Corollary 3.1 and Lemma 4.5.

We also mention the following useful result.

Lemma 4.6. *If* ξ *is semi–reduced, then* ξ *or* $\xi - 1$ *is also reduced in the regular continued fraction sense.*

Proof. By [A2, Theorem V, p. 30], we have $\xi > \frac{1+\sqrt{5}}{2} > 1$ and by [A2, Theorem III, p. 27], $-1 < \overline{\xi} < 1$. So if $-1 < \overline{\xi} < 0$, ξ is RCF–reduced. If $0 < \overline{\xi}$, let $\xi = \frac{P+\sqrt{D}}{Q}$. Then $0 < \frac{P-\sqrt{D}}{Q}$ and as $0 < Q < \sqrt{D}$ by [A2, Theorem 1(iv), p. 22], we deduce

$$2 < \frac{2\sqrt{D}}{Q} < \frac{P+\sqrt{D}}{Q} = \xi.$$

Hence $1 < \xi - 1$ and as $-1 < \overline{\xi - 1} < 0$, it follows that $\xi - 1$ is RCF–reduced. ☐

We conclude with an example of Ayyangar [A2, Theorem XIII, p. 103]. His proof of case (b), when $p > 2q$, involved a complicated discussion of inequalities.

Example Let $\xi = \frac{P+\sqrt{D}}{Q} = \frac{p+q+\sqrt{p^2+q^2}}{p}, p > 0, q > 0$. Then

(a) ξ is not reduced if $p < 2q$,

(b) ξ is reduced if $p \ge 2q$,

Proof. Here $P = p+q, Q = p, R = (D-P^2)/Q = -2q$ and

$$Q^2 + \tfrac{1}{4}R^2 = p^2 + q^2 = D \text{ and } \tfrac{1}{4}Q^2 + R^2 = \tfrac{1}{4}p^2 + 4q^2.$$

Hence

$$\begin{aligned} \tfrac{1}{4}Q^2 + R^2 \le D &\iff \tfrac{1}{4}p^2 + 4q^2 \le p^2 + q^2 \\ &\iff 3q^2 \le 3p^2/4 \\ &\iff 2q \le p. \end{aligned}$$

Consequently if $p < 2q$, ξ is not reduced. However if $p \geq 2q$, then $\frac{1}{4}Q^2 + R^2 \leq D$. Also the positive–negative representation

$$\begin{aligned} 1/\xi &= \frac{-P+\sqrt{D}}{R} = \frac{-p-q+\sqrt{p^2+q^2}}{-2q} \\ &= 0 + \frac{p}{p+q+\sqrt{p^2+q^2}} = 1 - \frac{p}{p-q+\sqrt{p^2+q^2}}, \end{aligned}$$

shows that ξ is the successor of $1/\xi$. Hence conditions (i) and (ii) of our Theorem are satisfied, so ξ is semi–reduced. Also condition (iii) is satisfied. For assume $\xi = \frac{P+Q+\sqrt{P^2+Q^2}}{2Q}$, $P > 2Q > 0$. Then

$$p + q = P + Q, \quad p = 2Q, \quad p^2 + q^2 = P^2 + Q^2.$$

Hence

$$\begin{aligned} P^2 + Q^2 &= 4Q^2 + q^2 \\ P^2 &= 3Q^2 + q^2 = 3Q^2 + (P-Q)^2 \\ &= 4Q^2 + P^2 - 2PQ \\ 2PQ &= 4Q^2 \\ P &= 2Q. \end{aligned}$$

This contradiction completes the demonstration that ξ is reduced if $p \geq 2q > 0$. □

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References

[A1] A. A. K. Ayyangar, *A new continued fraction*, Current Sci. **6** (1938), 602–604.

[A2] A. A. K. Ayyangar, *Theory of the nearest square continued fraction*, J. Mysore Univ. Sect. A. **1** (1941), 97–117.

[MR] K. R. Matthews and J. P. Robertson, *On the definition of nearest–integer reduced quadratic surd*, `http://www.numbertheory.org/pdfs/nicf-reduced.pdf`.

[P] O. Perron, *Die Lehre von den Kettenbrüchen*, Band 1, Teubner, Stuttgart, 1954.