On the definition of nearest-integer reduced quadratic surd

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Abstract

Hurwitz gave a definition of reduced quadratic surd for his nearest integer continued fraction expansion which characterises purely periodic surds. We give a more accessible proof of this equivalence.

Introduction. In Perron's book [1, pp.168-169], a *nearest integer* continued fraction (Kettenbrüche nächsten Ganzen) expansion of ξ_0 is defined recursively:

$$\xi_v = b_v + \frac{a_{v+1}}{\xi_{v+1}}, -\tfrac{1}{2} < \xi_v - b_v < \tfrac{1}{2}, \tag{1}$$

where $a_{v+1} = \pm 1$, b_v is an integer (the nearest integer to ξ_v) and $\mathrm{sign}(a_{v+1}) = \mathrm{sign}(\xi_v - b_v)$.

Thus

$$\xi_0 = b_0 + \frac{a_1|}{|b_1} + \cdots + \frac{a_k|}{|b_k} + \cdots \tag{2}$$

and we have

$$b_v \geq 2,\ b_v + a_{v+1} \geq 2 \text{ for } v \geq 1.$$

(Satz 10, [1, p.169]).

On [1, p.173], Perron defines a *singular* continued fraction (singuläre Kettenbrüche) as one satisfying

$$b_v \geq 2,\ b_v + a_v \geq 2 \text{ for } v \geq 1.$$

If expansion (2) is purely periodic with period k (ie. $a_{k+1} = a_1, b_k = b_0$ and $\xi_k = \xi_0$, then with $\eta_v = \overline{\xi_v}$, we have (see [1, p. 82]) the periodic singular continued fraction expansion

$$-\eta_0 = \frac{a_k|}{|b_{k-1}} + \frac{a_{k-1}|}{|b_{k-2}} + \cdots + \frac{a_1|}{|b_0} + \frac{a_k|}{|b_{k-1}} + \cdots \tag{3}$$

Now in Satz 14, [1, p.174], it is proved that the value of a singular continued fraction lies between $-\frac{3-\sqrt{5}}{2}$ and $\frac{\sqrt{5}-1}{2}$. Hence (3) gives

$$-\frac{3-\sqrt{5}}{2} \leq -\eta_0 \leq \frac{\sqrt{5}-1}{2},$$

or

$$\frac{1-\sqrt{5}}{2} \leq \eta_0 \leq \frac{3-\sqrt{5}}{2}. \tag{4}$$

In fact the left-hand inequality is strict, for equality implies $\xi = \frac{1+\sqrt{5}}{2}$, whose nearest integer continued fraction expansion

$$\frac{1+\sqrt{5}}{2} = 2 - \frac{1|}{|3} - \frac{1|}{|3} - \cdots$$

is not purely periodic. Moreover equality in the right-hand inequality occurs when $\xi_0 = \frac{3+\sqrt{5}}{2}$, which does have a purely periodic nearest integer continued fraction expansion

$$\frac{3+\sqrt{5}}{2} = 3 - \frac{1|}{|3} - \frac{1|}{|3} - \cdots .$$

Hence we have proved

Theorem 1. If the NICF expansion (2) is purely periodic, then either $\xi_0 = \frac{3+\sqrt{5}}{2}$ or $\xi_0 > 2$ and

$$\frac{1-\sqrt{5}}{2} < \eta_0 < \frac{3-\sqrt{5}}{2}. \tag{5}$$

We prove the converse of Theorem 1 by imitating the argument in [2, pp.384-385].

We assume we have a nearest integer continued fraction expansion (1) of ξ_0 where $\xi_0 > 2$ and inequalities (5) hold. We use induction and assume the corresponding inequalities hold for η_v:

$$\frac{1-\sqrt{5}}{2} < \eta_v < \frac{3-\sqrt{5}}{2}. \tag{6}$$

$$\frac{1-\sqrt{5}}{2} < b_v + \frac{a_{v+1}}{\overline{\xi_{v+1}}} < \frac{3-\sqrt{5}}{2}. \tag{7}$$

Case 1. $a_{v+1} = 1$. Then $b_v \geq 2$ and (7) give

$$\frac{1}{\overline{\xi_{v+1}}} < \frac{-1-\sqrt{5}}{2} < 0$$

$$0 > \overline{\xi_{v+1}} > \frac{-2}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{2}.$$

Case 2. $a_{v+1} = -1$. Then $\xi_v < b_v$, so $b_v \geq 3$. Then (7) gives

$$\frac{-1}{\overline{\xi_{v+1}}} < \frac{-3-\sqrt{5}}{2} < 0,$$

so $\overline{\xi_{v+1}} > 0$. Also

$$\frac{1}{\overline{\xi_{v+1}}} > \frac{3+\sqrt{5}}{2}$$

$$\overline{\xi_{v+1}} < \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}.$$

Hence the induction goes through.

By applying the pigeon-hole principle to the pairs (ξ_v, a_v), it follows that there are non-negative integers $i < j$ such that $\xi_i = \xi_j$ and $a_i = a_j$ and hence $-a_i/\overline{\xi_i} = -a_j/\overline{\xi_j}$.

But from (7) that with $r = \frac{3-\sqrt{5}}{2}$,

$$b_v = \lfloor r - \frac{a_{v+1}}{\overline{\xi_{v+1}}} \rfloor. \tag{8}$$

so we have $\xi_{i-1} = \xi_{j-1}$. Continuing this argument gives $\xi_0 = \xi_{j-i}$ and so the nearest integer expansion for ξ_0 is purely periodic.

References

[1] O. Perron, *Kettenbrüchen*, Chelsea Publishing Company, 1950.

[2] K. Rosen, *Introduction to number theory and its applications*, Addison-Wesley, 1984.