

# Primitive Pythagorean triples and the negative Pell equation

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## Abstract

**Abstract.** This paper uses continued fractions to give more explicit versions of results of A. Grytczuk, F. Luca and M. Wójtowicz and of K. Hardy and K.S. Williams relating the solvability of the negative Pell equation to the existence of primitive Pythagorean triples. These results were also obtained by P. Kaplan and K.S. Williams, with somewhat different proofs.

## 1 Introduction.

This note started on reading a short paper of Grytczuk, Luca and Wójtowicz (GLW) [2], which proved that the negative Pell equation  $x^2 - Dy^2 = -1$ ,  $D > 1$  and non-square, is solvable in positive integers  $x$  and  $y$  if and only if there exist a primitive Pythagorean triple  $(A, B, C)$  (ie.  $A, B, C$  are positive integers satisfying  $A^2 + B^2 = C^2$  and  $\gcd(A, B) = 1$ ) and positive integers  $a, b$  such that

$$D = a^2 + b^2 \text{ and } |aA - bB| = 1.$$

Then earlier related papers of K. Hardy and K.S. Williams [3] and P. Kaplan and K.S. Williams [5] came to the attention of the author.

The primitive Pythagorean triples  $(A, B, C)$  with  $A$  even, are given by

$$A = 2uv, B = u^2 - v^2, C = u^2 + v^2, \tag{1}$$

where  $u > v > 0$ ,  $\gcd(u, v) = 1$  and  $u$  and  $v$  have different parity.

Then the condition  $|aA - bB| = 1$  becomes  $|2auv - b(u^2 - v^2)| = 1$ . The solubility of this diophantine equation is equivalent to that of  $bV^2 - 2aVW - bW^2 = 1$  and criteria for solubility of this last equation were discussed in [3] and [5].

Sufficiency is immediate: If  $x = |aB + bA|$  and  $y = C$ , then

$$Dy^2 = (a^2 + b^2)(A^2 + B^2) = (aB + bA)^2 + (aA - bB)^2 = x^2 + 1. \quad (2)$$

(This was also noticed by Hardy and Williams in Theorem 2, [3, page 147].

It is easy to see that  $D$  is not a perfect square. (The proof given in Theorem 1 [3, page 146] is rather long.)

The authors GLW gave two proofs of the necessity part, one of these proofs being in terms of gcd's in  $Z[i]$ . Hardy and Williams also use this approach in their Theorem 3, page 148.

Kaplan and Williams also give a proof, using continued fractions - see Lemma 3, [5, p. 174-176]. We give a slightly different proof in Theorem 2.1 and show that if the negative Pell equation  $x^2 - Dy^2 = -1$  has a solution, then there is a pair  $(a, b)$  of relatively prime positive integers such that  $D = a^2 + b^2$ ,  $b$  odd and such that a primitive Pythagorean triple  $(A, B, C)$  exists with  $|aA - bB| = 1$  and  $A$  even.

In Theorem 4.1, we show conversely that  $(a, b)$  is unique and Theorem 2.1 gives all such  $(A, B, C)$ .

In practice, we find that if  $D \neq a^2 + 1$ , that  $(x, y) = (aB + bA, C)$  is often the fundamental solution of the negative Pell equation.

## 2 Producing primitive Pythagorean triples

**NOTATION 2.1** The continued fraction expansion of  $\sqrt{D}$  is  $[a_0, \overline{a_1, \dots, a_l}]$ , with period-length  $l$ . Let  $(P_i + \sqrt{D})/Q_i$  denote the  $i$ -th complete convergent and  $A_i/B_i$  the  $i$ -th convergent, where  $P_0 = 0, Q_0 = 1, A_{-2} = 0, A_{-1} = 1, B_{-2} = 1, B_{-1} = 0, a_0 = \lfloor \sqrt{D} \rfloor$  and for  $i \geq 1$ ,

- (a)  $P_i = a_{i-1}Q_{i-1} - P_{i-1}$ ,
- (b)  $Q_i = (D - P_i^2)/Q_{i-1}$ ,
- (c)  $a_i = \lfloor (P_i + \sqrt{D})/Q_i \rfloor$ .

It is well-known ([8, page 93] that the negative Pell equation is soluble if and only if the continued fraction expansion of  $\sqrt{D}$  has odd period-length  $l$ .

Suppose  $l = 2n - 1$ . Then the positive solutions  $(x, y)$  of  $x^2 - Dy^2 = -1$  are given by  $(x_t, y_t) = (A_{2N-2}, B_{2N-2})$ , where  $N = n + t(2n - 1), t \geq 0$ . In fact  $x_t + y_t\sqrt{D} = (x_0 + y_0\sqrt{D})^{2t+1}$ , where  $(x_0, y_0) = (A_{2n-2}, B_{2n-2})$  is the smallest (*fundamental*) positive solution of the negative Pell equation.

**THEOREM 2.1** Suppose  $\sqrt{D} = [a_0, \overline{a_1, \dots, a_l}]$ , where the period-length  $l = 2n - 1$  is odd. Let  $N = n + t(2n - 1), t \geq 0$  and  $u = B_{N-1}$  and  $v = B_{N-2}$ . Also let

$$a = P_n = P_N, b = Q_n = Q_N, A = 2uv, B = u^2 - v^2, C = u^2 + v^2.$$

Then

- (a)  $D = a^2 + b^2, b$  odd.
- (b)  $aA - bB = (-1)^N$ .
- (c)  $\gcd(u, v) = 1, u > v$  and one of  $u$  and  $v$  is even.
- (d)  $x_t = aB + bA, y_t = C$ .
- (e)  $A = (bx_t - \epsilon a)/D, B = (ax_t + \epsilon b)/D$ , where  $\epsilon = (-1)^{N-1}$ .

**REMARK 2.1** (i) We have  $A \geq 0, B > 0$ . Also  $A = 0 \Leftrightarrow n = 1 \Leftrightarrow D = \alpha^2 + 1$  for some  $\alpha \geq 1$ .

(ii) From (e), taking  $N = n$  and  $t = 0$ , we get congruences

$$bx_0 \equiv \epsilon a \pmod{D} \text{ and } ax_0 \equiv -\epsilon b \pmod{D}. \quad (3)$$

This result is also part of Theorem 5 [3, page 154] where it is stated that the congruence  $x_0e \equiv d \pmod{D}$  has precisely four coprime integer solutions  $(e, d)$  satisfying  $|d| < \sqrt{D}$  and  $|e| < \sqrt{D}$ . These are by (3)  $\pm(\epsilon b, a)$  and  $\pm(a, -\epsilon b)$ .

### 3 Some lemmas

**LEMMA 3.1** *Let  $\sqrt{D}$  have period-length  $l$ . Then for  $i \geq 1$ ,*

$$DB_{il-1} = a_0 A_{il-1} + A_{il-2} \quad (4)$$

$$A_{il-1} = a_0 B_{il-1} + B_{il-2}. \quad (5)$$

**Proof.** See equations (16) and (17) [8, p. 70] for the case  $i = 1$ . The proof goes over to a general period  $il$ .

**LEMMA 3.2** *With  $N = n + t(2n - 1)$  as in Theorem 1.1,*

$$DB_{2N-2} = A_{N-1}^2 + A_{N-2}^2 \quad (6)$$

$$A_{2N-2} = A_{N-1}B_{N-1} + A_{N-2}B_{N-2}$$

$$B_{2N-2} = B_{N-1}^2 + B_{N-2}^2. \quad (7)$$

**Proof of Lemma 3.2.** We start from the matrix identity

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{2N-2} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_{2N-2} & A_{2N-3} \\ B_{2N-2} & B_{2N-3} \end{bmatrix} \quad (8)$$

and partition the above matrix product as

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{N-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_N & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{2N-2} & 1 \\ 1 & 0 \end{bmatrix}.$$

But  $a_{N+i} = a_{N-i-1}$  for  $i = 0, \dots, N-2$ , so (8) becomes

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{N-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{N-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_{2N-2} & A_{2N-3} \\ B_{2N-2} & B_{2N-3} \end{bmatrix}.$$

Multiplying both sides of this equation on the right by  $\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$  then gives

$$\begin{aligned} \begin{bmatrix} A_{N-1} & A_{N-2} \\ B_{N-1} & B_{N-2} \end{bmatrix} \begin{bmatrix} A_{N-1} & A_{N-2} \\ B_{N-1} & B_{N-2} \end{bmatrix}^t &= \begin{bmatrix} a_0 A_{2N-2} + A_{2N-3} & A_{2N-2} \\ a_0 B_{2N-2} + B_{2N-3} & B_{2N-2} \end{bmatrix} \\ &= \begin{bmatrix} DB_{2N-2} & A_{2N-2} \\ A_{2N-2} & B_{2N-2} \end{bmatrix}, \end{aligned} \quad (9)$$

by Lemma 3.1 with  $i = (2t + 1)$ ,  $l = 2n - 1$ ,  $2N - 2 = il - 1$ . Hence

$$\begin{bmatrix} A_{N-1} & A_{N-2} \\ B_{N-1} & B_{N-2} \end{bmatrix} \begin{bmatrix} A_{N-1} & B_{N-1} \\ A_{N-2} & B_{N-2} \end{bmatrix} = \begin{bmatrix} DB_{2N-2} & A_{2N-2} \\ A_{2N-2} & B_{2N-2} \end{bmatrix}. \quad (10)$$

Finally, equation (10) implies equations (6) and (7).

**LEMMA 3.3**

$$A_{i-1} = Q_i B_{i-2} + B_{i-1} P_i \text{ for } i \geq 0, \quad (11)$$

$$A_{N-2} = Q_N B_{N-1} - P_N B_{N-2}, \quad (12)$$

**Proof.** For (11) see [8, p. 70].

For (12), we note that  $Q_N = Q_{N-1}$ . Then

$$\begin{aligned} Q_N B_{N-1} - P_N B_{N-2} &= Q_N (a_{N-1} B_{N-2} + B_{N-3}) \\ &\quad - (a_{N-1} Q_{N-1} - P_{N-1}) B_{N-2} \\ &= Q_N B_{N-3} + P_{N-1} B_{N-2} \\ &= Q_{N-1} B_{N-3} + P_{N-1} B_{N-2} = A_{N-2}, \end{aligned}$$

by equation (11), with  $i = N - 1$ .

**Proof of Theorem 2.1.**

Part (a) is proved in [8, p. 95] and also in [11], where it is pointed out that  $b$  is odd.

(b)

$$\begin{aligned} aA - bB &= P_n(2B_{N-1}B_{N-2}) - Q_n(B_{N-1}^2 - B_{N-2}^2) \\ &= P_N(2B_{N-1}B_{N-2}) - Q_N(B_{N-1}^2 - B_{N-2}^2) \\ &= B_{N-1}(P_N B_{N-2} - B_{N-1} Q_N) + \\ &\quad + B_{N-2}(Q_N B_{N-2} + B_{N-1} P_N) \\ &= B_{N-1}(-A_{N-2}) + B_{N-2} A_{N-1} \\ &= (-1)^N. \end{aligned} \quad (13)$$

(c)  $u = B_{N-1} \geq v = B_{N-2}$  always holds. However equality would imply  $2auv = (-1)^N$ . Also  $\gcd(B_{N-1}, B_{N-2}) = 1$  follows from equation (13) above.

(d)

$$\begin{aligned} A^2 + B^2 &= (2B_{N-1}B_{N-2})^2 + (B_{N-1}^2 - B_{N-2}^2)^2 \\ &= (B_{N-1}^2 + B_{N-2}^2)^2 \\ &= B_{2N-2}^2 \text{ by (7)} \\ &= C^2. \end{aligned}$$

Next,

$$\begin{aligned}
aB + bA &= P_N(B_{N-1}^2 - B_{N-2}^2) + Q_N(2B_{N-1}B_{N-2}) \\
&= B_{N-1}(P_N B_{N-1} + Q_N B_{N-2}) + B_{N-2}(Q_N B_{N-1} - P_N B_{N-2}) \\
&= B_{N-1}A_{N-1} + B_{N-2}A_{N-2} = A_{2N-2} \text{ by (6)}.
\end{aligned}$$

(e)

$$\begin{aligned}
b^2y^2 &= b^2B^2 + b^2A^2 \\
&= b^2B^2 + (x - aB)^2 \\
&= x^2 - 2aBx + DB^2.
\end{aligned}$$

Hence  $DB^2 - 2aBx + x^2 - b^2y^2 = 0$ .

But  $x^2 = Dy^2 - 1$ , so  $DB^2 - 2aBx + a^2y^2 - 1 = 0$ . Hence

$$\begin{aligned}
B &= \frac{ax \pm \sqrt{a^2x^2 - D(a^2y^2 - 1)}}{D} \\
&= \frac{ax \pm \sqrt{a^2(Dy^2 - 1) - D(a^2y^2 - 1)}}{D} \\
&= \frac{ax \pm \sqrt{b^2}}{D} = \frac{ax + \eta b}{D}, \eta = \pm 1.
\end{aligned}$$

Next

$$A = \frac{x - a\left(\frac{ax + \eta b}{D}\right)}{b} = \frac{b^2x - \eta ab}{bD} = \frac{bx - \eta a}{D}.$$

Finally

$$\begin{aligned}
aA - bB &= \frac{a(bx - \eta a)}{D} - \frac{b(ax + \eta b)}{D} \\
&= \frac{-\eta(a^2 + b^2)}{D} = (-1)^N.
\end{aligned}$$

Hence  $\eta = (-1)^{N-1}$ .

**EXAMPLE 3.1** (Also given in [1, page 243]).

Let  $c$  and  $t$  be positive integers,  $k$  a non-negative integer and  $c^2 + 1 \equiv 0 \pmod{t^2}$ . Also let  $D = \frac{c^2+1}{t^2} + 2ck + t^2k^2$ . Then

(a)

$$\sqrt{D} = \left[ \left[ \frac{c}{t} \right] + tk, a_1, \dots, a_{l-1}, 2\left[ \frac{c}{t} \right] + 2tk \right],$$

where  $c/t = \left[ \left[ \frac{c}{t} \right], a_1, \dots, a_{l-1} \right]$ .

(b) If  $k \geq 1$ , then  $(c + t^2k, t)$  is the fundamental solution  $(X, Y)$  of the negative Pell equation  $x^2 - Dy^2 = -1$ . Also  $l$  is odd,  $l = 2n - 1$ .

(c)  $P_n = \alpha k + \beta, Q_n = \gamma k + \delta$ , where

$$\alpha = B, \beta = (Bc + (-1)^n A)/t^2, \gamma = A, \delta = (Ac - (-1)^n B)/t^2.$$

Here  $A = 2uv, B = u^2 - v^2$ , and  $(u, v) = (B_{n-1}, B_{n-2})$ , where these refer simultaneously to the continued fraction expansion of  $c/t$  (with a last partial quotient  $m$  replaced by  $m - 1, 1$ , (if necessary) to get a palindromic list) and  $\sqrt{D}$ , while  $P_n$  and  $Q_n$  refer to that of  $\sqrt{D}$ . Moreover  $Q_n$  is odd (ie.  $\delta$  is odd),  $\gcd(P_n, Q_n) = 1 = \gcd(\alpha, \gamma) = \gcd(\beta, \delta)$ .

(d)  $D = \frac{c^2+1}{t^2} + 2ck + t^2k^2 = (\alpha k + \beta)^2 + (\gamma k + \delta)^2$ .

(e)  $-2(\alpha k + \beta)uv + (\gamma k + \delta)(u^2 - v^2) = (-1)^n$ .

**Proof.** (a) is proved in Theorem 4.1 [10, p. 74] and (a) and (b) in Theorem 1 (a) [7, pp. 231-232].

(b) follows from the fact that  $(x, y) = (c + t^2k, t)$  is a solution of  $x^2 - Dy^2 = -1$  and that if  $(x, y)$  is a positive solution of  $x^2 - Dy^2 = -1$ , not the fundamental solution  $(X, Y)$ , then as  $(x, y) = (X, Y)^{2i+1}, i \geq 1$ , then  $y \geq D$  and hence  $y > t^2$  would then hold if  $k \geq 1$ .

To prove (c), using the notation for  $a = P_n$  and  $b = Q_n$  of Theorem 2.1, from parts (b) and (d) with  $N = n$ , we have equations

$$a\alpha + b\gamma = c + t^2k \tag{14}$$

$$a\gamma - b\alpha = (-1)^n \tag{15}$$

which can be solved for  $a$  and  $b$ .

Finally, (d) and (c) follow from Theorem (2.1), parts (a) and (b).

## 4 Uniqueness of $a$ and $b$

We give a version of the "only if" part of Theorem 3 of [3, p. 148] and Lemma 2 [5, pp. 171-174], which characterises  $a$  and  $b$  in terms of continued fractions and which in view of Theorem (2.1), shows that there is exactly one pair  $(a, b)$  for which (16) below is soluble.

**THEOREM 4.1** . Suppose  $A = 2uv, B = u^2 - v^2$  with  $u > v > 0$  and  $\gcd(u, v) = 1$ , with one of  $u$  and  $v$  even. Also  $a$  and  $b$  are positive integers satisfying

$$|aA - bB| = 1. \quad (16)$$

Then with  $D = a^2 + b^2$ , we have

- (a)  $x = aB + bA, y = C = u^2 + v^2$  satisfies  $x^2 - Dy^2 = -1$ ,
- (b)  $a = P_n$  and  $b = Q_n$ , where  $2n - 1$  is the period-length of the continued fraction expansion of  $\sqrt{D}$ .
- (c)  $u = B_{N-1}, v = B_{N-2}, x = A_{2N-2}, y = B_{2N-2}$ , where  $N = n + t(2n - 1)$ . Also  $aA - bB = (-1)^N$ .

**REMARK 4.1** Hardy and Williams characterise  $a$  and  $b$  instead in terms of gcd's in  $\mathbb{Z}[i]$  in Theorem 3 [3, p. 148].

**Proof.** We need the following result, which is Theorem 172 of [4, pp. 140-141] in slightly more general form:

**LEMMA 4.1** . Let  $\omega = \frac{P\zeta + R}{Q\zeta + S}$ , where  $\zeta > 1$  and  $P, Q, R, S$  are integers such that  $Q > 0, S > 0$  and  $PS - QR = \pm 1$ . Then  $P/Q$  is a convergent  $A_k/B_k$  to  $\omega$ . Moreover, if  $Q > S$ , then  $R/S = A_{k-1}/B_{k-1}, k \geq 0$ . Also  $\zeta$  is the  $(k + 1)$ -th complete convergent to  $\omega$ .

In Lemma 4.1 take  $P = (au + bv), R = bu - av, Q = u, S = v, \zeta = (a + \sqrt{D})/b$ . Then

$$\sqrt{D} = (P\zeta + Q)/(R\zeta + S),$$

where  $PS - QR = \pm 1$ . Also  $\zeta > 1$  and  $Q > S > 0$ .

For

$$\begin{aligned}
(P\zeta + Q)/(R\zeta + S) &= \frac{(au + bv)\frac{(a+\sqrt{D})}{b} + (bu - av)}{u\frac{(a+\sqrt{D})}{b} + v} \\
&= \frac{(au + bv)(a + \sqrt{D}) + b(bu - av)}{u(a + \sqrt{D}) + bv} \\
&= \frac{(a^2u + b^2u + (au + bv)\sqrt{D})}{ua + bv + u\sqrt{D}} \\
&= \frac{(Du + (au + bv)\sqrt{D})}{ua + bv + u\sqrt{D}} \\
&= \sqrt{D}.
\end{aligned}$$

Also

$$\begin{aligned}
PS - QR &= (au + bv)v - u(bu - av) \\
&= (auv + bv^2) - (bu^2 - auv) \\
&= 2auv - b(u^2 - v^2) = \pm 1,
\end{aligned}$$

from equation (16).

It follows from Lemma 4.1 that

$$P/Q = (au + bv)/u = A_{N-1}/B_{N-1}, \quad (17)$$

$$R/S = (bu - av)/v = A_{N-2}/B_{N-2}, \quad (18)$$

$$(a + \sqrt{D})/b = (P_N + \sqrt{D})/Q_N, \quad (19)$$

for some  $N \geq 1$ .

Hence as  $\gcd(au + bv, u) = 1 = \gcd(bu - av, v)$ , we have from (17) and (19)

$$au + bv = A_{N-1}, \quad u = B_{N-1}, \quad (20)$$

$$bu - av = A_{N-2}, \quad v = B_{N-2}. \quad (21)$$

Also from (19) we have

$$P_N = a \text{ and } Q_N = b. \quad (22)$$

Now let  $\epsilon = 2auv - b(u^2 - v^2)$ . Then

$$\epsilon = (-1)^N, b = Q_{N-1}. \quad (23)$$

$$\begin{aligned}
(-1)^{N-1} &= A_{N-2}B_{N-1} - A_{N-1}B_{N-2} \\
&= (bu - av)u - (au + bv)v \\
&= -2auv + b(u^2 - v^2) = -\epsilon.
\end{aligned}$$

$$\begin{aligned}
(-1)^{N-1}Q_{N-1} &= A_{N-2}^2 - DB_{N-2}^2 \\
&= (bu - av)^2 - (a^2 + b^2)v^2 \\
&= -2abuv + b^2(u^2 - v^2) = (-1)^{N-1}b.
\end{aligned}$$

Finally, let  $2n - 1$  be the period-length of  $\sqrt{D}$ . Then as  $Q_{N-1} = Q_N$ , it follows that  $N \equiv n \pmod{2n - 1}$ .

Note that by periodicity,  $a = P_N = P_n$  and  $b = Q_N = Q_n$ .

We can now express  $x$  and  $y$  in terms of the convergents  $A_{N-1}/B_{N-1}$  and  $A_{N-2}/B_{N-2}$ :

$$\begin{aligned}
x &= aB + bA = a(u^2 - v^2) + b(2uv) \\
&= u(au + bv) + v(bu - av) \\
&= B_{N-1}A_{N-1} + B_{N-2}A_{N-2} = A_{2N-2}. \tag{24}
\end{aligned}$$

$$y = C = u^2 + v^2 = B_{N-1}^2 + B_{N-2}^2 = B_{2N-2}. \tag{25}$$

## 5 Examples

**EXAMPLE 5.1**  $D = 13$ .  $\sqrt{13} = [3, \overline{1, 1, 1, 1, 6}]$ ,  $l = 5, n = 3$ .  $a = P_3 = 2, b = Q_3 = 3, \eta = (18, 5)$ . Then with  $N = n + tl = 3 + 5t, t \geq 0$ ,  $(u, v) = (B_{N-1}, B_{N-2})$ ,  $(A, B, C) = (2uv, u^2 - v^2, u^2 + v^2)$ ,  $(x_t, y_t) = (aB + bA, C)$ , we have

$t$	$(u, v)$	$(A, B, C)$	$aA - bB$	$(x_t, y_t)$
0	(2, 1)	(4, 3, 5)	-1	$\eta$
1	(71, 38)	(5396, 3597, 6485)	1	$\eta^3$
2	(2558, 1369)	(7003804, 4669203, 8417525)	-1	$\eta^5$

**EXAMPLE 5.2** (The case  $b = 1, a \geq 1$ .) In Theorem (2.1) let  $D = a^2 + 1, a \geq 1$ . Then  $\sqrt{D} = [a, \overline{2a}]$  and  $l = 1 = n, P_1 = a, Q_1 = 1$ .

Then with  $N = 1 + t, t \geq 0$  and  $(u, v) = (B_t, B_{t-1})$ , where  $B_{-1} = 0, B_0 = 1, B_i = 2aB_{i-1} + B_{i-2}, i \geq 1$ , we have  $2auv - (u^2 - v^2) = (-1)^{t+1}, \gcd(u, v) = 1$  and  $u > v$ .

Also with  $A = 2uv, B = u^2 - v^2, C = u^2 + v^2$ , we have  $|aA - B| = 1$ .

The fundamental solution of  $x^2 - Dy^2 = -1$  is  $\eta = (a, 1)$  and  $(x, y) = (aB + bA, C) = \eta^{2t+1}, t \geq 0$ .

**EXAMPLE 5.3** The case  $a = 1$  and  $b > 1$  cannot occur. ie. the equation  $|-bu^2 + 2uv + bv^2| = 1$  has no integer solutions if  $b > 1$ .

**Proof.** Assume  $a = 1$  and  $b > 1$ . Then (16) becomes

$$|-bu^2 + 2uv + bv^2| = 1.$$

Consider the matrix

$$H = \begin{bmatrix} u & u + bv \\ v & bu - v \end{bmatrix}$$

Then  $\det H = -\epsilon$  and all entries are positive.

Also if  $D = b^2 + 1$ ,

$$\omega = \frac{1 + \sqrt{D}}{b} = \frac{u\sqrt{D} + u + bv}{v\sqrt{D} + bu - v}.$$

Hence by Lemma 4.1,  $u/v$  is a convergent  $A_{k-1}/B_{k-1}$  to  $\omega$ .

Now  $\omega = \overline{[1, b-1, 1]}$ .

Also by Theorem 5.3.4 [6, p. 246],

$$bA_{k-1}^2 - 2A_{k-1}B_{k-1} - bB_{k-1}^2 = (-1)^k Q_k,$$

where  $(P_k + \sqrt{D})/Q_k$  is the  $k$ -th complete convergent to  $\omega$ . Hence

$$\pm 1 = bu^2 - 2uv - bv^2 = (-1)^k Q_k, \quad (26)$$

But we readily verify that for  $i \geq 0$ ,

1. (a)  $(P_{3i} + \sqrt{D})/Q_{3i} = (1 + \sqrt{D})/b$ ,
2. (b)  $(P_{3i+1} + \sqrt{D})/Q_{3i+1} = (b - 1 + \sqrt{D})/2$ ,
3. (c)  $(P_{3i+2} + \sqrt{D})/Q_{3i+2} = (b - 1 + \sqrt{D})/b$ .

Hence equation (26) gives a contradiction.

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