An improvement of the Minkowski bound for real quadratic orders using the Markoff theorem

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Abstract

Using the Markoff theorem on indefinite binary quadratic forms it is shown that every element of the class group of any real quadratic order of discriminant d has an ideal of norm less than or equal to $1 + \lfloor \frac{\sqrt{d}}{3} \rfloor$.

1 Introduction

Let $d \equiv 0, 1 \pmod 4$ be a positive integer that is not a square. Let R denote the unique quadratic order of discriminant d in $\mathbb{Q}(\sqrt{d})$. It is well known that every element of the class group of ideals in R has an ideal whose norm is less than $\frac{\sqrt{d}}{2}$, the Minkowski bound ([5, Theorem 1.3.1]). This bound can be improved to $\sqrt{\frac{d}{5}}$ ([3, Theorem 11, page 141]). In the following theorem we present a bound that is best possible.

Theorem 1.1 Every element in the class group of a real quadratic order of discriminant d has an ideal of norm less that or equal to $1 + \lfloor \frac{\sqrt{d}}{3} \rfloor$.

The bound given in the above theorem is best possible because there are quadratic orders of discriminant d which have ideal classes where the least norm is $1 + \lfloor \frac{\sqrt{d}}{3} \rfloor$.

The proof of the above theorem uses the remarkable theorem of Markoff which states that if f is an indefinite binary quadratic form of discriminant

d, such that the least positive integer represented by it is greater than $\sqrt{d}/3$, then f is equivalent to a multiple of a Markoff form.

There are a countable number of Markoff forms where each such form corresponds to a solution triple (a, b, c) with $0 < a \leq b \leq c$ of the Markoff equation $x^2 + y^2 + z^2 = 3xyz$. The Markoff conjecture states that given a positive integer c, there is at most one pair of positive integers a, b with $a \leq b \leq c$ such that (a, b, c) is a solution triple of the Markoff equation.

Our proof of Theorem 1.1 is based on the standard construction of Markoff forms as in [1], where a Markoff form is defined for each Markoff number c and denoted by F_c. There is an ambiguity in this notation as the Markoff form defined actually depends on the ordered triple (a, b, c) and not only on c. Therefore if the Markoff conjecture is not true, then given a Markoff number c, it is possible that there is more than one such form F_c. To avoid this ambiguity we define a Markoff form for each ordered solution triple (a, b, c) of the Markoff equation. Moreover our construction also enables us to state a conjecture that is equivalent to the Markoff conjecture as follows.

We show that every Markoff form corresponds to a pair of binary quadratic forms f, f^{-1} of discriminant $9c^2 - 4$, where f represents c and f^2 represents -1. Therefore the Markoff conjecture is equivalent to the following statement: There is at most one pair of forms f, f^{-1} in the form class group of discriminant $9c^2 - 4$ such that f represents the integer c and f^2 represents -1.

2 Binary quadratic forms and ideals

2.1 Forms

A *binary quadratic form* $f = (a, b, c)$ is a function $f(x, y) = ax^2 + bxy + cy^2$, where a, b, c are real numbers called the *coefficients* and $d = b^2 - 4ac$ is called the *discriminant*. In the case when the form is *integral*, that is a, b, c are integers we will assume that $\gcd(a, b, c) = 1$. Sometimes we will suppress the last coefficient and write simply (a, b) instead of (a, b, c) as the value of c can be computed from the discriminant equation.

Two forms f and f' are said to be *equivalent*, written as $f \sim f'$, if for some $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ we have $f'(x, y) = f(\alpha x + \beta y, \gamma x + \delta y)$. It is easy to see that $\sim$ is an equivalence relation on the set of forms of discriminant d.

If $f = (a, b, c)$, then the form $(a, -b, c)$ is denoted as f^{-1} and the form $(-a, b, -c)$ is denoted as $\overline{f}$. Note that in general f^{-1} does not represent an inverse; only integral forms have inverses as explained below. A form g is a *multiple* of form f if there exists a real number k such that $g = kf = (ka, kb, kc)$. Observe that $-f = (\overline{f})^{-1}$.

A form f is said to *represent* an integer m if there exist coprime integers x and y, such that $f(x, y) = m$. Note that equivalent forms represent the same integers and hence sometimes we refer to a class of forms f that represents a given integer. Also it is easy to see that f and f^{-1} represent the same set of integers.

In the case of integral forms, the set of equivalence classes of forms is an abelian group called the *form class group* with group law as composition given in Definition 2.1 in the next section. The *class number* denoted by $h(d)$ is the order of the class group.

The *inverse* of an integral form f in the class group is f^{-1} defined above and the *identity form* e is defined as the form $(1, 0, \frac{-d}{4})$ or $(1, 1, \frac{1-d}{4})$ depending on whether d is even or odd, respectively. Note that any form that represents the integer 1 is equivalent to the identity form.

The *infimum* of a binary quadratic form f is defined as $m(f) = \inf\{|f(x, y)| : x, y \in \mathbb{Z}\}$, where x, y are not both 0. Note that $m(f) = m(f^{-1}) = m(-f)$.

The results in the following lemma are well known; the reader may refer to [6] for proofs.

Lemma 2.1

1. The form (a, b) is equivalent to the form $(a, b + 2a\delta)$ for any integer δ.

2. A form f represents an integer n if and only if $f \sim (n, b, c)$ for some numbers b, c.

3. Let a prime p be represented by an integral form f. If g is any form that represents p, then either $g \sim f$ or $g \sim f^{-1}$.

The following lemma can be proved easily using standard results. Part 1 follows because the fundamental unit in this case has positive norm and Part 2 follows from [7, Lemma 4.4].

Lemma 2.2 Let $d = 9c^2 - 4$. Then the following hold in the form class group.

1. The identity form e is not equivalent to $\overline{e}$.

2. If $|n| < \sqrt{d}/2$ and the identity form e represents n, then $n = 1$.

2.2 Ideals

In this section all forms are integral and $d \equiv 0, 1 \pmod 4$ denotes a positive integer that is not a square. R denotes the unique quadratic order of discriminant d in $K = \mathbb{Q}(\sqrt{d})$. For details of the results presented in this section the reader is directed to [2, Section 5.2 and 5.4.2]. A clear presentation of the arithmetic of ideals is also available in [5, Sections 1.2 and 1.3].

A primitive integral ideal I of R can be written in the form

$$I = a\mathbb{Z} + \frac{-b+\sqrt{d}}{2}\mathbb{Z}, \tag{2.1}$$

where a, b are integers such that $a > 0$ is the norm of the ideal and $4a$ divides $b^2 - d$.

If $c = \frac{b^2-d}{4a}$ and $\gcd(a, b, c) = 1$, then the ideal is invertible and the inverse is the ideal $\overline{I} = a\mathbb{Z} + \frac{b+\sqrt{d}}{2}\mathbb{Z}$. Note that $(a, b, c) = ax^2 + bxy + cy^2$ is a form of discriminant d. Indeed the invertible ideals are the ideals that correspond to forms.

Let F be the set of forms modulo the action of the group $\left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, m \in \mathbb{Z} \right\}$, where the forms (a, b, c) and $(a, b + 2am, c')$ are identified. Let $\mathbb{I}$ be the set of fractional ideals modulo $\mathbb{Q}^*$. A map ψ from F to $\mathbb{I}$ is defined as

$$\psi(a, b, c) = \begin{cases} a\mathbb{Z} + \frac{-b+\sqrt{d}}{2}\mathbb{Z}, & \text{if } a > 0 \\ (a\mathbb{Z} + \frac{-b+\sqrt{d}}{2}\mathbb{Z})\sqrt{d}, & \text{if } a < 0. \end{cases}$$

The map ψ induces a bijection between the class group of forms and the *narrow class group* of R, where two ideals I and J are strictly equivalent, written as $I \approx J$, if there are algebraic integers α and β such that $\alpha I = \beta J$ and the norms of α and β are of the same sign. To establish the bijection induced by ψ it is necessary to consider for each ideal an ordered basis $w_1\mathbb{Z} + w_2\mathbb{Z}$ that satisfies $\overline{w_2}w_1 - \overline{w_1}w_2 > 0$, where $\overline{w_1}$ represents the conjugate of w_1. Two ideals are said to be equivalent, written as $I \sim J$, if the ideal equality given above holds without the norm condition. The group of equivalence classes of ideals under this equivalence is called the *wide* class group or simply *class group*.

Observe that if $a > 0$ and $\psi(f) = \psi(a, b, c) = I$, then $\psi(\overline{f}) = \psi(-a, b, -c) = I\sqrt{d}$. Moreover, in the (wide) class group, we have the ideal equivalence $\psi(f) \sim \psi(\overline{f})$.

In the following definition we present the formula for the product of ideals which leads to composition of forms.

Let $I_k = a_k\mathbb{Z} + \frac{-b_k+\sqrt{d}}{2}\mathbb{Z}$, $k = 1, 2$, be two primitive ideals. Let $f_1 = (a_1, b_1, c_1)$ and $f_2 = (a_2, b_2, c_2)$ be the corresponding binary quadratic forms of discriminant d.

Definition 2.1 Let $g = gcd(a_1, a_2, (b_1 + b_2)/2)$ and let v_1, v_2, w be integers such that

$$v_1a_1 + v_2a_2 + w(b_1 + b_2)/2 = g.$$

If we define a_3 and b_3 as

$$a_3 = \frac{a_1a_2}{g^2},$$
$$b_3 = b_2 + 2\,\frac{a_2}{g}\,\left(\frac{b_1 - b_2}{2}\;v_2 - c_2w\right),$$

then $I_1 \cdot I_2$ is the ideal $a_3\mathbb{Z} + \frac{-b_3+\sqrt{d}}{2}\mathbb{Z}$. Further the composition of the forms (a_1, b_1, c_1) and (a_2, b_2, c_2) is the form (a_3, b_3, c_3), where c_3 is computed using the discriminant equation.

Note that this gives the multiplication in the class group.

3 The Markoff theory

A triple (a, b, c) of positive integers that satisfies the Markoff equation

$$a^2 + b^2 + c^2 = 3abc \tag{3.2}$$

is called a Markoff triple; the numbers a, b, and c are called Markoff numbers. We call a solution triple (a, b, c) ordered if $a \le b \le c$.

We define below a Markoff form for which we first observe that if (a, b, c) is a Markoff triple and k is an integer such that $ak \equiv b(\text{mod } c)$, then from the Markoff equation (3.2) it follows that $k^2 + 1 \equiv 0(\text{mod } c)$.

Definition 3.1 Let (a,b,c) be an ordered Markoff triple. Let integer k be defined as

$$ak \equiv b \pmod{c}, \quad 0 \le k \le c,$$

and let l be defined by

$$k^2 + 1 = lc.$$

Then the Markoff form F_{abc} corresponding to the triple (a,b,c), is defined as

$$F_{abc} = \left(1, 3 - \frac{2k}{c}, \frac{l-3k}{c}\right).$$

Note that

$$cF_{abc} = (c, 3c - 2k, l - 3k)$$

is an integral form. Moreover $(cF_{abc})^{-1} \sim (c, 3c-2k', l'-3k')$ where $k' = c-k$ with $k'^2 + 1 = l'c$ so that while $k \equiv \frac{b}{a} \pmod{c}$ we have $k' \equiv \frac{a}{b} \pmod{c}$.

Lemma 3.1 Let F_{abc} be a Markoff form and $f = cF_{abc}$. Then

1. $F_{abc} \sim -F_{abc}$ or equivalently $f^2 \sim \overline{e}$, that is $f \sim -f$.
2. The infimum $m(F_{abc}) = 1$ or equivalently $m(f) = c$.

Proof. See [1, Chapter 2, Lemmas 9 and 10] □

For the next lemma we need the following definitions. For an ordered Markoff triple (c_1, c_2, c) let k_1, k_2, k be defined as

$$\begin{cases} k \equiv \frac{c_2}{c_1} \pmod{c}, & 0 \le k < c, \\ k_1 \equiv \frac{c}{c_2} \pmod{c}_1, & 0 \le k_1 < c_1, \\ k_2 \equiv \frac{c_1}{c} \pmod{c}_2, & 0 \le k_2 < c_2. \end{cases}$$

Lemma 3.2 Let (c_1, c_2, c) be an ordered Markoff triple and let k_1, k_2, k be as defined above. Then

1. $(3c_1c_2 - c, c_1, c_2)$, $(c_1, c, 3cc_1 - c_2)$ and $(c_2, c, 3cc_2 - c_1)$ are ordered Markoff triples.
2. $F_{abc}(k, c) = 1$ and $F_{abc}(k_2 - 3c_2, c_2) = F_{abc}(k_1, c_1) = -1$ or equivalently $cF_{abc}(k, c) = c$ and $cF_{abc}((k_2 - 3c_2, c_2) = cF_{abc}(k_1, c_1) = -c$.

Proof. See [1, Chapter 2, page 28 and Lemma 8]. □

Lemma 3.3 Let F_{abc} and $F_{a'b'c}$ be two Markoff forms corresponding to two distinct ordered Markoff triples (a,b,c) and (a',b',c). Then the forms cF_{abc} and $cF_{a'b'c}$ are inequivalent forms of discriminant $d = 9c^2 - 4$.

Proof. Let $f = cF_{abc} = (c, 3c-2k, l-3k)$ where k, l are as given in Definition 3.1. Similarly let $f' = cF_{a'b'c} = (c, 3c-2k', l'-3k')$. Then $k^2 \equiv (k')^2 \pmod{c}$. It is known that every even Markoff number c satisfies $c \equiv 2 \pmod{4}$ (see for instance [7, Lemma 4.2]). Therefore we have a factorization $c = c_1c_2$ where $\gcd(c_1, c_2) = 1$ and

$$k \equiv k' \pmod{c}_1, \quad k \equiv -k' \pmod{c}_2. \tag{3.3}$$

If $c_1 \leq c_2$ using Definition 2.1 for composition of forms, we have $ff' = (a_3, b_3, c_3)$, where $a_3 = c^2/g = c_1^2$ as $g = \gcd(c, k+k') = c_2$. If $f \sim f'$, then $ff' \sim f^2 \sim \overline{e}$ by Lemma 3.1 and hence $\overline{e}$ represents $a_3 = c_1^2 < c$ (Lemma 2.1). In the case when $c_2 \leq c_1$ we consider the composition of f with f'^{-1} to get $ff'^{-1} = (a_3, b_3, c_3)$, where $a_3 = c^2/g = c_2^2$ as $g = \gcd(c, k-k') = c_1$. Therefore in this case if $f \sim f'$, then $ff'^{-1} = e$ represents $a_3 = c_2^2 < c$. In either case e represents an integer a_3 such that $|a_3| < c < \sqrt{d}/2$. From Lemma 2.2 it follows that $|a_3| = 1$ which means from (3.3) that either c_1 or c_2 is equal to 1. If c_2 is 1, then $k = k'$ and so $f = f'$. If $c_1 = 1$ then $k \equiv -k' \pmod{c}$, which means $f' \sim f^{-1}$ which combined with $f \sim f'$ gives $f^2 \sim e$ which is not true as $f^2 \sim \overline{e}$ by Lemma 3.1 and by Lemma 2.2 the forms e and $\overline{e}$ are not equivalent. □

Lemma 3.4 Let f be a form of discriminant $9c^2 - 4$ that represents c and such that $f^2 \sim \overline{e}$. Then either f or f^{-1} is equivalent to cF_{abc} for some ordered Markoff triple (a,b,c).

Proof. Firstly from Lemma 2.1 parts 1 and 2, as f represents c we have $f \sim (c, 3c-2k, l-3k)$ for some integers l, k with $1 \leq k \leq c$ and

$$k^2 + 1 = cl. \tag{3.4}$$

As $f^2 \sim \overline{e}$, we have $f \sim -f$, which means that f represents $-c$. Let $y = c_1$ be the least positive value of y of all representations $f(x,y) = -c$. Then for some integer k_1 with $\gcd(k_1, c_1) = 1$ we have $f(k_1, c_1) = -c$, that is

$$ck_1^2 + (3c-2k)k_1c_1 + (l-3k)c_1^2 = -c. \tag{3.5}$$

Let

$$c_1k - ck_1 = c_2. \tag{3.6}$$

We will show that (c_1, c_2, c) is a Markoff triple. We have using (3.4)-(3.6)

$$\begin{aligned} c_1^2 + c_2^2 + c^2 &= c_1^2 + c_1^2k^2 + c^2k_1^2 - 2cc_1kk_1 + c^2 \\ &= c^2 + c_1^2(k^2+1) + c^2k_1^2 - 2cc_1kk_1 \\ &= c^2 + c_1^2lc + c^2k_1^2 - 2cc_1kk_1 \\ &= c(c + c_1^2l + ck_1^2 - 2c_1kk_1) \\ &= c(3kc_1^2 - 3ck_1c_1) = 3cc_1(c_1k - ck_1) \\ &= 3cc_1c_2. \end{aligned} \tag{3.7}$$

By way of choice of c_1, it follows from Lemma 3.2 that $c = \max\{c_1, c_2, c\}$. If $c_1 \leq c_2$ then (c_1, c_2, c) is an ordered triple and from Definition 3.1 and (3.6) it follows that $f \sim F_{c_1c_2c}$. If $c_2 < c_1$, then from the remark following Definition 3.1 we have $f \sim (cF_{c_2c_1c})^{-1}$. □

Markoff in [4] proved the following result using continued fractions. Cassels [1, Theorem II, page 39] gave a proof using Markoff forms. Cassels defines Markoff forms for triples (a, b, c) where c is greater than both a and b. We define Markoff forms for ordered triples and hence make the necessary adjustment in the statement of the theorem below.

Theorem 3.1 (Markoff) Suppose that $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ with $\delta = \beta^2 - 4\alpha\gamma > 0$. Then if $m(f) > \sqrt{d}/3$ then f or f^{-1} is equivalent to a multiple of a Markoff form.

We conclude this section with a conjecture that is equivalent to the Markoff conjecture.

Conjecture 3.1 Let $d = 9c^2 - 4$. Then there is at most one pair of classes of forms of discriminant d, $\{f, f^{-1}\}$, that represents c and such that $f^2 = f^{-2} = \overline{e}$.

Theorem 3.2 Conjecture 3.1 is equivalent to the Markoff conjecture.

Proof. The equivalence of the two conjectures follows from the fact that there is a one to one correspondence between ordered Markoff triples (a, b, c) and pairs of forms $\{f, f^{-1}\}$ that represent c and such that $f^2 = \overline{e}$. This correspondence is clear from Definition 3.1, Lemma 3.1, Lemma 3.3 and Lemma 3.4. □

4 Proof of Theorem 1.1 and corollaries

Let C denote the form class group of forms of discriminant $d > 0$. Recall that C is isomorphic to the narrow class group of ideals in R via the bijection ψ given in Section 2.2. If $N(I)$ is the norm of the ideal I, then note that the infimum of a form is given by $m(f) = inf\{N(I) : I \sim \psi(f)\}$.

Lemma 4.1 Let f be an integral form of discriminant $d > 0$ such that $m(f) > \frac{\sqrt{d}}{3}$. Then one of the following holds.

1. $d = 9c^2 - 4$ where c is an odd Markoff number and $m(f) = c$ with $f \sim g$ or $f^{-1} \sim g$ where $g = cF_{abc}$ for some ordered Markoff triple (a, b, c).
2. $d = 9c_1^2 - 1$ where $c = 2c_1$ is an even Markoff number and $m(f) = c_1$ with $f \sim g$ or $f^{-1} \sim g$ where $g = c_1F_{abc}$ for some ordered Markoff triple (a, b, c).

Moreover if f' is any form of discriminant d with $m(f') > \frac{\sqrt{d}}{3}$, then $m(f') = m(f)$.

Proof. From Theorem 3.1 one of f or f^{-1}, say f, is equivalent to $g = rF_{abc}$ for some real number r and Markoff form F_{abc}. By Lemma 3.1 part 1, as $F_{abc} \sim -F_{abc}$, we may assume that $r > 0$. From Definition 3.1 we have

$$f \sim g = \left(r, 3r - \frac{2kr}{c}, \frac{(l-3k)r}{c}\right).$$

As $m(f) = m(g) = rm(F_{abc}) = r$ (by Lemma 3.1), r is an integer and we have on comparing discriminants of f and g,

$$dc^2 = r^2(9c^2 - 4). \tag{4.8}$$

If c is odd then $\gcd(c, 9c^2 - 4) = 1$ and hence $c|r$. As f is a primitive form g also is primitive and hence $r = c$ as r/c divides the coefficients of g. Therefore $d = 9c^2 - 4$ and the result follows in this case.

In the case when c is even we have $c = 2c_1$ and hence from (4.8) it follows that $c_1|r$. In this case r/c_1 divides the coefficients of g and hence as g is a primitive form we have $r = c_1$ and $d = 9c_1^2 - 1$.

Now let f' be a form with $m(f') > \frac{\sqrt{d}}{3}$. Then from the above we have if $m(f) \neq m(f')$ then $d = 9c^2 - 4 = 9c_1^2 - 1$ with $m(f) = c$ and $m(f') = c_1$. However this is not possible as the equality does not hold modulo 9. □

Proof of Theorem 1.1

Let f be a binary quadratic form of discriminant d and suppose that an ideal class $\psi(f)$ has no ideals of norm $\leq 1+\lfloor\frac{\sqrt{d}}{3}\rfloor$. It follows that $m(f) > \frac{\sqrt{d}}{3}$. By Lemma 4.1 above $d = 9m(f)^2 - 4$ or $d = 9m(f)^2 - 1$. In either case we have $m(f) = 1 + \lfloor\frac{\sqrt{d}}{3}\rfloor$. Now f represents $m(f)$ and hence $\psi(f)$ has an ideal of norm $m(f) = 1 + \lfloor\frac{\sqrt{d}}{3}\rfloor$, a contradiction and the result follows. □

Corollary 4.1 Let d be a fundamental discriminant. If all primes $p \leq 1 + \lfloor\frac{\sqrt{d}}{3}\rfloor$ are inert primes, that is $(\frac{d}{p}) = -1$, then the class number $h(d) = 1$.

Proof. If d is a fundamental discriminant, then it is well known that the class group is generated by ideals of prime norm. If $\psi(f)$ is an ideal class then from Theorem 1.1 we have $m(f) \leq 1 + \lfloor\frac{\sqrt{d}}{3}\rfloor$. Now every prime p dividing $m(f)$ is less than or equal to $1 + \lfloor\frac{\sqrt{d}}{3}\rfloor$ and satisfies $(\frac{d}{p}) \neq -1$ and hence $m(f) = 1$ by assumption which gives $h(d) = 1$. □

Several results in the literature use the Minkowski bound $\sqrt{d}/2$ or $\sqrt{\frac{d}{5}}$ as an upper bound for the norms of ideals that generate the class group. For example in [5, Theorem 6.2.1, page 200] Mollin improves a previous result by replacing the Minkowski bound by $\sqrt{\frac{d}{5}}$. Given Theorem 1.1, we may now further improve this result by using $1 + \lfloor\frac{\sqrt{d}}{3}\rfloor$ instead of $\sqrt{\frac{d}{5}}$:

Corollary 4.2 Let d be a fundamental discriminant and $q_i \geq 1$ for $1 \leq i \leq n$ be pairwise relatively prime square free divisors of d. Let $F(x) = qx^2 + (\alpha - 1)qx + ((\alpha - 1)q^2 - d)/(4q)$, where $\alpha = 1$ if d is even and $\alpha = 2$ otherwise. Then, if for every prime $p \leq 1+\lfloor\frac{\sqrt{d}}{3}\rfloor$ with $(\frac{d}{p}) \neq -1$ and $p \neq q_i$ for any i, there is a $q = \prod_{i\in S} q_i$ for some $S \subset \{1, 2, \cdots n\}$ such that $|F(x)| = p$ for some $x \geq 0$, then the class group is generated by the R-ideals over q_i for $i = 1, 2, \cdots n$. In other words the class group is generated by ideals of order 2.

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