ON SHANKS' ALGORITHM FOR COMPUTING THE CONTINUED FRACTION OF $\log_b a$.

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ABSTRACT. We give a more practical variant of Shanks' 1954 algorithm for computing the continued fraction of $\log_b a$, for integers $a > b > 1$, using the floor and ceiling functions and an integer parameter $c > 1$. The variant, when repeated for a few values of $c = 10^r$, enables one to guess if $\log_b a$ is rational and to find approximately r partial quotients.

1. SHANKS' ALGORITHM

In his article [1], Shanks gave an algorithm for computing the partial quotients of $\log_b a$, where $a > b$ are positive integers greater than 1. Construct two sequences $a_0 = a, a_1 = b, a_2, \ldots$ and $n_0, n_1, n_2, \ldots$, where the a_i are positive rationals and the n_i are positive integers, by the following rule: If $i \geq 1$ and $a_{i-1} > a_i > 1$, then

$$a_i^{n_{i-1}} \leq a_{i-1} < a_i^{n_{i-1}+1} \tag{1.1}$$

$$a_{i+1} = a_{i-1}/a_i^{n_{i-1}}. \tag{1.2}$$

Clearly (1.1) and (1.2) imply $a_i > a_{i+1} \geq 1$. Also (1.1) implies $a_i \leq a_{i-1}^{1/n_{i-1}}$ for $i \geq 1$ and hence by induction on $i \geq 0$,

$$a_{i+1} \leq a_0^{1/n_0 \cdots n_i}. \tag{1.3}$$

Also by induction on $j \geq 0$,

$$a_{2j} = a_0^r/a_1^s, \quad a_{2j+1} = a_1^u/a_0^v, \tag{1.4}$$

where r and u are positive integers and s and v are non–negative integers.

Two possibilities arise:

(i) $a_{r+1} = 1$ for some $r \geq 1$. Then equations (1.4) imply a relation $a_0^q = a_1^p$ for positive integers p and q and so $\log_{a_1} a_0 = p/q$.

(ii) $a_{i+1} > 1$ for all i. In this case the decreasing sequence $\{a_i\}$ tends to $a \geq 1$. Also (1.3) implies $a = 1$, unless perhaps $n_i = 1$ for all sufficiently large i; but then (1.2) becomes $a_{i+1} = a_{i-1}/a_i$ and hence $a = a/a = 1$.

If $a_{i-1} > a_i > 1$, then from (1.1) we have

$$n_{i-1} = \left\lfloor \frac{\log a_{i-1}}{\log a_i} \right\rfloor. \tag{1.5}$$

Let $x_i = \log_{a_{i+1}} a_i$ if $a_{i+1} > 1$. Then we have

Lemma 1. *If* $a_{i+2} > 1$, *then*

$$x_i = n_i + 1/x_{i+1}. \tag{1.6}$$

1991 *Mathematics Subject Classification.* Primary 11D09.

Proof. From (1.2), we have

$$\log a_{i+2} = \log a_i - n_i \log a_{i+1} \tag{1.7}$$

$$1 = \frac{\log a_i}{\log a_{i+1}} \cdot \frac{\log a_{i+1}}{\log a_{i+2}} - n_i \cdot \frac{\log a_{i+1}}{\log a_{i+2}} \tag{1.8}$$

$$= x_i x_{i+1} - n_i x_{i+1}, \tag{1.9}$$

from which (1.6) follows. □

From Lemma 1.1 and (1.5), we deduce

Lemma 2. (a) *If $\log_{a_1} a_0$ is irrational, then*

$$x_i = n_i + 1/x_{i+1} \text{ for all } i \geq 0.$$

(b) *If $\log_{a_1} a_0$ is rational, with $a_{r+1} = 1$, then*

$$x_i = \begin{cases} n_i + 1/x_{i+1} & \text{if } 0 \leq i < r-1, \\ n_{r-1} & \text{if } i = r-1. \end{cases}$$

In view of the equation $\log_{a_1} a_0 = x_0$, Lemma 2 leads immediately to

Corollary 1.

$$\log_{a_1} a_0 = \begin{cases} [n_0, n_1, \ldots] & \text{if } \log_{a_1} a_0 \text{ is irrational,} \\ [n_0, n_1, \ldots, n_{r-1}] & \text{if } \log_{a_1} a_0 \text{ is rational and } a_{r+1} = 1. \end{cases} \tag{1.10}$$

Remark. It is an easy exercise to show that for $j \geq 0$,

$$a_{2j} = a_0^{q_{2j-2}} / a_1^{p_{2j-2}}, \quad a_{2j+1} = a_1^{p_{2j-1}} a_0^{q_{2j-1}} \tag{1.11}$$

where p_k/q_k is the k–th convergent to $\log_{a_1} a_0$.

Example 1. $\log_2 10$: Here $a_0 = 10$, $a_1 = 2$. Then $2^3 < 10 < 2^4$, so $n_0 = 3$ and $a_2 = 10/2^3 = 1.25$.

Further, $1.25^3 < 2 < 1.25^4$, so $n_1 = 3$ and $a_3 = 2/1.25^3 = 1.024$.

Also, $1.024^9 < 1.25 < 1.024^{10}$, so $n_2 = 9$ and

$$\begin{aligned} a_4 &= 1.25/1.024^9 \\ &= 1250000000000000000000000000/1237940039285380274899124224 \\ &= 1.0097419586\cdots \end{aligned}$$

Continuing we obtain Table 1 and $\log_2 10 = [3, 3, 9, 2, 2, 4, 6, 2, 1, 1, \ldots]$.

Table 1

i	n_i	a_i	p_i/q_i
0	3	10	3/1
1	3	2	10/3
2	9	1.25	93/28
3	2	1.024	196/59
4	2	$1.0097419586\cdots$	485/146
5	4	$1.0043362776\cdots$	2136/643
6	6	$1.0010415475\cdots$	13301/4004
7	2	$1.0001628941\cdots$	28738/8651
8	1	$1.0000637223\cdots$	42039/12655
9	1	$1.0000354408\cdots$	70777/21306
10		$1.0000282805\cdots$	
11		$1.0000071601\cdots$	

2. Some Pseudocode

In Table 2 we present pseudocode for the Shanks algorithm.

It soon becomes impractical to perform the calculations in multiprecision arithmetic, as the numerators and denominators a_i grow rapidly. If we truncate the decimal expansions of the `a[i]` to r places and represent a positive rational a as $g(a)/10^r$, where $g(a) = \lfloor 10^r a \rfloor$, the ratio `aa/bb` will be calculated as $\lfloor 10^r g(aa)/g(bb) \rfloor$. Working explicitly in integers, using the $g(a)$, then results in algorithm 1, also depicted in Table 2, with $c = 10^r$, where `int(x,y)` equals $\lfloor x/y \rfloor$, when x and y are integers.

As shown in the next section, the `A[i]` decrease strictly until they reach `c`. Also `m[0]`=`n[0]` and we can expect a number of the initial `m[i]` will be partial quotients. Naturally, the larger we take c, the more partial quotients will be produced.

3. Formal description of algorithm 1

We show in Theorem 2.1 below, that algorithm 1 will give the correct partial quotients when $\log_{a_1} a_0$ is rational and otherwise gives a parameterised sequence of integers which tend to the correct partial quotients when $\log_{a_1} a_0$ is irrational.

Algorithm 1 is now explicitly described. We define two integer sequences $\{A_{i,c}\}$, $i = 0, \ldots, l(c)$ and $\{m_{j,c}\}$, $j = 0, \ldots, l(c) - 2$, as follows.

Let $A_{0,c} = c \cdot a_0, A_{1,c} = c \cdot a_1$. Then if $i \geq 1$ and $A_{i-1,c} > A_{i,c} > c$, we define $m_{i-1,c}$ and $A_{i+1,c}$ by means of an intermediate sequence $\{B_{i,r,c}\}$, defined for $r \geq 0$, by $B_{i,0,c} = A_{i-1,c}$ and

$$B_{i,r+1,c} = \left\lfloor \frac{cB_{i,r,c}}{A_{i,c}} \right\rfloor, r \geq 0. \tag{3.1}$$

Then $c \leq B_{i,r+1,c} < B_{i,r,c}$, if $B_{i,r,c} \geq A_{i,c} > c$ and hence there is a unique integer $m = m_{i-1,c} \geq 1$ such that

$$B_{i,m,c} < A_{i,c} \leq B_{i,m-1,c}.$$

Then we define $A_{i+1,c} = B_{i,m,c}$. Hence $A_{i+1,c} \geq c$ and the sequence $\{A_{i,c}\}$ decreases strictly until $A_{l(c),c} = c$.

There are two possible outcomes, depending on whether or not $\log_b(a)$ is rational:

TABLE 2

Shanks' algorithm	algorithm 1
`input: integers a>b>1`	`input: integers a>b>1, c> 1`
`output: n[0],n[1],...`	`output: m[0],m[1],...`
`s:= 0`	`s:= 0`
`a[0]:= a; a[1]:= b`	`A[0]:= a*c; A[1]:= b*c`
`aa:= a[0]; bb:= a[1]`	`aa:= A[0]; bb:= A[1]`
`while(bb > 1){`	`while(bb > c){`
`  i:=0`	`  i:=0`
`  while(aa ≥ bb){`	`  while(aa ≥ bb){`
`    aa:= aa/bb`	`    aa:= int(aa*c,bb)`
`    i:= i+1`	`    i:= i+1`
`  }`	`  }`
`  a[s+2]:= aa`	`  A[s+2]:= aa`
`  n[s]:= i`	`  m[s]:= i`
`  t:= bb`	`  t:= bb`
`  bb:= aa`	`  bb:= aa`
`  aa:= t`	`  aa:= t`
`  s:= s+1`	`  s:= s+1`
`}`	`}`

Theorem 2. (1) $\log_{a_1} a_0$ *is a rational number* p/q, $p > q \geq 1$, $\gcd(p,q) = 1$. *Then*

(a) $a_0 = d^p$, $a_1 = d^q$ *for some positive integer* d*;*

(b) *if* $p/q = [n_0, \dots, n_{r-1}]$*, where* $n_{r-1} > 1$ *if* $r > 1$*, then*

(i) $A_{r+1,c} = c, a_{r+1} = 1$*;*

(ii) $A_{i,c} = c \cdot a_i$ *for* $0 \leq i \leq r+1$*;*

(iii) $m_{i,c} = n_i$ *for* $0 \leq i \leq r-1$.

(2) $\log_{a_1} a_0$ *is irrational. Then*

(a) $m_{0,c} = n_0$*;*

(b) $l(c) \to \infty$ *and for fixed* i*,* $A_{i,c}/c \to a_i$ *as* $c \to \infty$ *and* $m_{i,c} = n_i$ *for all large* c.

Proof. 1(a) follows from the equation $a_1^p = a_0^q$.

1(b) is also straightforward on noticing that a_i is a power of d and that we are implicitly performing Euclid's algorithm on the pair (p,q).

For 2(a), we have

$$a_1^{n_0} < a_0 < a_1^{n_0+1} \tag{3.2}$$

and $A_{0,c} = c \cdot a_0$, $A_{1,c} = c \cdot a_1$. Also by induction on $0 \leq r \leq n_0$,

$$B_{1,r,c} \geq ca_1^{n_0-r}, \tag{3.3}$$

$$B_{1,r,c} \leq \frac{ca_0}{a_1^r}. \tag{3.4}$$

Inequality (3.3) with $r \leq n_0 - 1$ gives $B_{1,r,c} \geq A_{1,c}$, while inequality (3.4) with $r = n_0$ gives

$$B_{1,n_0,c} \leq \frac{ca_0}{a_1^{n_0}} < ca_1 = A_{1,c},$$

by inequality (3.2). Hence $m_{0,c} = n_0$.
For 2(b), we use induction on $i \geq 1$ and assume $l(c) \geq i$ holds for all large c and that $A_{i-1,c}/c \to a_{i-1}$ and $A_{i,c}/c \to a_i$ as $c \to \infty$. This is clearly true when $i = 1$.

By properties of the integer part symbol, equation (3.1) gives

$$\frac{c^r A_{i-1,c}}{A^r_{i,c}} - \frac{(1 - \frac{c^r}{A^r_{i,c}})}{1 - \frac{c}{A_{i,c}}} < B_{i,r,c} \leq \frac{c^r A_{i-1,c}}{A^r_{i,c}}. \tag{3.5}$$

for $r \geq 0$.

Hence for $r < n_{i-1}$, inequalities (3.5) give

$$B_{i,r,c}/c \to a_{i-1}/a_i^r \geq a_{i-1}/a_i^{n_{i-1}-1} > a_i.$$

Then, because $A_{i,c}/c \to a_i$, it follows that $B_{i,r,c} > A_{i,c}$ for all large c.

Also $B_{i,n_{i-1},c}/c \to a_{i-1}/a_i^{n_{i-1}} < a_i$, so $B_{i,n_{i-1},c} < A_{i,c}$ for all large c. Hence $m_{i-1,c} = n_{i-1}$ for all large c. Also $A_{i+1,c} = B_{i,n_{i-1},c} > c$, so $l(c) > i+1$ for all large c. Moreover $A_{i+1,c}/c \to a_{i-1}/a_i^{n_{i-1}} = a_{i+1}$ and the induction goes through. □

Example 3. Table 3 lists the sequences $m_{0,c}, \ldots, m_{l(c)-2,c}$ for $c = 2^u, u = 1, \ldots, 30$, when $a_0 = 3, a_1 = 2$.

TABLE 3

```
1,1,
1,1,1,
1,1,1,1,
1,1,1,2,
1,1,1,2,
1,1,1,2,3,
1,1,1,2,2,2,
1,1,1,2,2,2,1,
1,1,1,2,2,2,1,2,
1,1,1,2,2,3,2,3,
1,1,1,2,2,3,2,
1,1,1,2,2,3,1,2, 1, 1,1, 2,
1,1,1,2,2,3,1,3, 1, 1,3, 1,
1,1,1,2,2,3,1,4, 3, 1,
1,1,1,2,2,3,1,4, 1, 9,1,
1,1,1,2,2,3,1,5,24, 1,2,
1,1,1,2,2,3,1,5, 3, 1,1, 2,7,
1,1,1,2,2,3,1,5, 2, 1,1, 5,3, 1,
1,1,1,2,2,3,1,5, 2, 2,1, 3,1,16,
1,1,1,2,2,3,1,5, 2,15,1, 6,2
1,1,1,2,2,3,1,5, 2, 9,5, 1,2,
1,1,1,2,2,3,1,5, 2,13,1, 1,1, 6,  1, 2, 2,
1,1,1,2,2,3,1,5, 2,17,2, 7,8,
1,1,1,2,2,3,1,5, 2,19,1,49,2, 1,
1,1,1,2,2,3,1,5, 2,22,4, 8,3, 4,  1,
1,1,1,2,2,3,1,5, 2,22,2, 1,3, 1,  3, 8,
1,1,1,2,2,3,1,5, 2,22,1, 6,3, 1,  1, 3, 4,  2,
1,1,1,2,2,3,1,5, 2,23,2, 1,1, 2,  1,12,17,
1,1,1,2,2,3,1,5, 2,23,3, 2,2, 2,  2, 1, 3,  2,
1,1,1,2,2,3,1,5, 2,23,2, 1,7, 2,  2,14, 1,  1, 6,
```

In fact $\log_2 3 = [1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, \ldots]$.

4. A heuristic algorithm

We can replace the $\lfloor x \rfloor$ function in equation (3.1) by $\lceil x \rceil$, the least integer exceeding x.

This produces an algorithm with similar properties to algorithm 1, with integer sequences $\{A'_{i,c}\}$, $i = 0, \ldots, l'(c)$ and $\{m'_{j,c}\}$, $j = 0, \ldots, l'(c) - 2$. Here $A_{0,c} = A'_{0,c} = a_0 \cdot c$, $A_{1,c} = A'_{1,c} = a_1 \cdot c$ and $m_{0,c} = m'_{0,c} = n_0$. Then if $i \geq 1$ and $A'_{i-1,c} > A'_{i,c} > c$, we define $m'_{i-1,c}$ and $A'_{i+1,c}$ by means of an intermediate sequence $\{B'_{i,r,c}\}$, defined for $r \geq 0$, by $B'_{i,0,c} = A'_{i-1,c}$ and

$$B'_{i,r+1,c} = \left\lceil \frac{cB'_{i,r,c}}{A'_{i,c}} \right\rceil, r \geq 0. \tag{4.1}$$

Then $c \leq B'_{i,r+1,c} < B'_{i,r,c}$, if $B'_{i,r,c} \geq A'_{i,c} > c$.

For

$$B'_{i,r+1,c} \leq \frac{cB'_{i,r,c}}{A'_{i,c}} + 1$$

and

$$\frac{cB'_{i,r,c}}{A'_{i,c}} + 1 \leq B'_{i,r,c} \quad \Leftrightarrow \quad cB'_{i,r,c} + A'_{i,c} \leq A'_{i,c}B'_{i,r,c}$$

$$\Leftrightarrow \quad \frac{A'_{i,c}}{A'_{i,c} - c} \leq B'_{i,r,c}.$$

The last inequality is certainly true if $B'_{i,r,c} \geq A'_{i,c} > c$.

Hence there is a unique integer $m' = m'_{i-1,c} \geq 1$ such that

$$B'_{i,m',c} < A'_{i,c} \leq B'_{i,m'-1,c}.$$

Then we define $A'_{i+1,c} = B'_{i,m',c}$. Hence $A'_{i+1,c} \geq c$ and the sequence $\{A'_{i,c}\}$ decreases strictly until $A'_{l'(c),c} = c$.

If we perform the two computations simultaneously, the common initial elements of the sequences $\{m_{j,c}\}$ and $\{m'_{k,c}\}$ are likely to be partial quotients of $\log_b(a)$. With $c = 10^r$ we expect roughly r partial quotients to be produced.

If $l(c) = l'(c)$ and $A_{j,c} = A'_{j,c}$ and $m_{j,c} = m'_{j,c}$ for $j = 0, \ldots, l(c) - 2$, then $\log_b a$ is likely to be rational.

In practice, to get a feeling of certainty regarding the output when $c = 10^r$, we also run the algorithm for $c = 10^t, r - 5 \leq t \leq r + 5$.

Example 4. Table 4 lists the common values of $m_{i,c}$ and $m'_{i,c}$, when $a = 3, b = 2$ and $c = 2^r, 1 \leq r \leq 31$. It seems likely that only partial quotients are produced for all $r \geq 1$.

Example 5. Table 5 lists the common values of $m_{i,c}$ and $m'_{i,c}$, when $a = 34, b = 2$ and $c = 10^r, 1 \leq r \leq 20$. Partial quotients are not always produced, as is seen from lines 9,14 and 17.

5. Acknowledgement

The second author is grateful for the hospitality provided by the School of Mathematical Sciences, ANU, where research for part of this paper was carried out.

TABLE 4. $a = 3, b = 2, c = 2^r, 1 \le r \le 31$.

```
 1: 1
 2: 1
 3: 1,1,1
 4: 1,1,1
 5: 1,1,1,2
 6: 1,1,1,2
 7: 1,1,1,2,2
 8: 1,1,1,2,2
 9: 1,1,1,2,2
10: 1,1,1,2,2
11: 1,1,1,2,2
12: 1,1,1,2,2
13: 1,1,1,2,2,3,1
14: 1,1,1,2,2,3,1
15: 1,1,1,2,2,3,1
16: 1,1,1,2,2,3,1,5
17: 1,1,1,2,2,3,1,5
18: 1,1,1,2,2,3,1,5
19: 1,1,1,2,2,3,1,5,2
20: 1,1,1,2,2,3,1,5
21: 1,1,1,2,2,3,1,5,2
22: 1,1,1,2,2,3,1,5,2
23: 1,1,1,2,2,3,1,5,2
24: 1,1,1,2,2,3,1,5,2
25: 1,1,1,2,2,3,1,5,2
26: 1,1,1,2,2,3,1,5,2
27: 1,1,1,2,2,3,1,5,2
28: 1,1,1,2,2,3,1,5,2,23
29: 1,1,1,2,2,3,1,5,2,23
30: 1,1,1,2,2,3,1,5,2,23,2
31: 1,1,1,2,2,3,1,5,2,23,2
```

TABLE 5. $a = 34, b = 12, c = 10^r, r = 1, \ldots, 20$.

```
 1: 1,2,2
 2: 1,2,2,1,1
 3: 1,2,2,1,1,2
 4: 1,2,2,1,1,2
 5: 1,2,2,1,1,2,3,1
 6: 1,2,2,1,1,2,3,1,8,1
 7: 1,2,2,1,1,2,3,1,8,1,1
 8: 1,2,2,1,1,2,3,1,8,1,1,2
 9: 1,2,2,1,1,2,3,1,8,1,1,2,2,1,13,3,2,32,7
10:1,2,2,1,1,2,3,1,8,1,1,2,2,1
11:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1
12:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1
13:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13
14:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,3
15:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2
16:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2
17:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,18,1,1,1,1,1
18:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
19:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
20:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
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