Some remarks about the solutions of $x^2 - Dy^2 = \pm N$.

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Proposition

(a) Let $U + V\sqrt{D} = (A + B\sqrt{D})(u + v\sqrt{D}$, where

$$A^2 - DB^2 = N, u^2 - Dv^2 = \pm 1, B > 0, u > 0, v > 0 \text{ and } B \leq |V|.$$

Then

(i) $A \geq 0 \Rightarrow U > 0$ and $V > 0$,

(ii) $A < 0$ and $N > 0 \Rightarrow U < 0$ and $V < 0$,
eg. $(-4 + \sqrt{3})(2 + \sqrt{3}) = -5 - 2\sqrt{3}$. $(u^2 - Dv^2 = 1)$;
$(-4 + \sqrt{13})(18 + 5\sqrt{13}) = -7 - 2\sqrt{13}$. $(u^2 - Dv^2 = -1)$;

(iii) $A < 0$ and $N < 0 \Rightarrow U > 0$ and $V > 0$.
eg. $(-1 + \sqrt{3})(2 + \sqrt{3}) = 1 + \sqrt{3}$. $(u^2 - Dv^2 = 1)$;
eg. $(-4 + 3\sqrt{13})(18 + 5\sqrt{13}) = 123 + 34\sqrt{13}$. $(u^2 - Dv^2 = -1)$;

(b) Let $U + V\sqrt{D} = (A + B\sqrt{D})(u - v\sqrt{D}$, where

$$A^2 - DB^2 = N, u^2 - Dv^2 = \pm 1, B > 0, u > 0, v > 0 \text{ and } B \leq |V|.$$

Then

(i) $A < 0 \Rightarrow U < 0 < V$,

(ii) $A \geq 0$ and $N > 0 \Rightarrow V < 0 < U$,
eg. $(4 + \sqrt{3})(2 - \sqrt{3}) = 5 - 2\sqrt{3}$. $(u^2 - Dv^2 = 1)$;
eg. $(4 + \sqrt{13})(18 - 5\sqrt{13}) = 7 - 2\sqrt{13}$. $(u^2 - Dv^2 = -1)$;

(iii) $A \geq 0$ and $N < 0 \Rightarrow U < 0 < V$.
eg. $(1 + \sqrt{3})(2 - \sqrt{3}) = -1 + \sqrt{3}$. $(u^2 - Dv^2 = 1)$;
eg. $(4 + 3\sqrt{13})(18 - 5\sqrt{13}) = -123 + 34\sqrt{13}$. $(u^2 - Dv^2 = -1)$;

Remark. (a) and (b) (i) are obvious. (b) (ii) and (iii) follow by conjugation from (a)(ii) and (iii).

Proof. We first prove (a)(ii). Assume $A < 0$.

$$U = -|A|u + BDv \quad (1)$$
$$V = -|A|v + Bu. \quad (2)$$

First assume $u^2 - Dv^2 = 1$.

Now $u > \sqrt{D}v$ and $|A| > \sqrt{D}B$. Hence $|A|u > BDv$ and by equation (1) we have $U < 0$.

Next $|A| \leq |U|$ as $B \leq |V|$ and $A^2 = DB^2 + N, U^2 = DV^2 + N$. Hence

$$\begin{aligned} |A| &\leq |A|u - BDv \\ BDv &\leq |A|(u-1) \\ BDvu &\leq |A|(u-1)u \end{aligned}$$

However

$$|A|(u-1)u < |A|v^2D \iff (u-1)u < v^2D \iff 1 = u^2 - Dv^2 < u.$$

Hence $BDvu < |A|v^2D$ and so $Bu < |A|v$. Then equation (2) implies $V < 0$.

Now assume $u^2 - Dv^2 = -1$.

We have $u < \sqrt{D}v$ and $\sqrt{D}B < |A|$. Hence $u\sqrt{D}B < |A|\sqrt{D}v$ and $uB < |A|v$.

Hence from equation (2), $V < 0$.

Also $B \leq |V| = |A|v - Bu$. Hence $B(1+u) < |A|v$.

We want to prove $|A|u > DBv$ ie. $|A| > DBv/u$.

But $|A| > B(1+u)/v$, so it suffices to prove

$$B(1+u)/v \geq DBv/u,$$

or $u \geq Dv^2 - u^2 = 1$.

Proof of (a)(iii). Assume $A < 0$.

First assume $u^2 - Dv^2 = 1$ and $A^2 - DB^2 = -|N|$. Then $u > \sqrt{D}v$ and $|A| < \sqrt{D}B$. Hence $|A|v < Bu$ and $V > 0$.

Next, we have to show $BDv > |A|u$. Suppose instead that $BDv \leq |A|u$. Now $B \leq |V|$ and (2) give $B \leq Bu - |A|v$. Hence

$$\begin{aligned} B(u-1) &\geq |A|v \\ B(u-1)/v &\geq |A|. \end{aligned}$$

Hence $u(B(u-1)/v \geq BDv$ and we deduce that $1 \geq u$, a contradiction.

Secondly, assume $u^2 - Dv^2 = -1$ and $A^2 - DB^2 = -|N|$. Now $u < \sqrt{D}v$ and $|A| < \sqrt{D}B$. Hence $u|A| < DBv$ and equation (2) gives $U > 0$.

We prove $Bu > |A|v$ by contradiction. Suppose $Bu \leq |A|v$. Then $B \leq |V| = |A|v - Bu$ and

$$\begin{array}{rcl} B(1+u) & \leq & |A|v \\ B(1+u)/v & \leq A & < \sqrt{D}B \\ (1+u)/v & < & \sqrt{D}. \end{array}$$

Hence $(1+u)^2 < Dv^2 = u^2 + 1$ and we have $2u < 0$, a contradiction.

Hence $Bu > |A|v$ and hence $V > 0$.

Corollary Let $x_0+y_0\sqrt{D}$ be a fundamental solution for a class of solutions to $x^2 - Dy^2 = N$. Also let η be the fundamental solution of the Pell's equation $x^2 - Dy^2 = 1$ and let

$$(x_0 + y_0\sqrt{D})\eta^n = x_n + y_n\sqrt{D}.$$

Then

(a) Suppose $N > 0$.

  (i) Suppose $x_0 > 0$. Then if $n > 0$, we have $x_n > 0$ and $y_n > 0$, while if $n < 0$, we have $x_n > 0$ and $y_n < 0$.

  (ii) Suppose $x_0 < 0$. Then if $n > 0$, we have $x_n < 0$ and $y_n < 0$, while if $n < 0$, we have $x_n < 0$ and $y_n > 0$.

(b) Suppose $N < 0$.

  (i) Suppose $x_0 > 0$. Then if $n > 0$, we have $x_n > 0$ and $y_n > 0$, while if $n < 0$, we have $x_n < 0$ and $y_n > 0$.

  (ii) Suppose $x_0 < 0$. Then if $n > 0$, we have $x_n > 0$ and $y_n > 0$, while if $n < 0$, we have $x_n < 0$ and $y_n > 0$.