Some remarks about the solutions of \( x^2 - Dy^2 = \pm N \).

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Proposition

(a) Let \( U + V \sqrt{D} = (A + B \sqrt{D})(u + v \sqrt{D}) \), where

\[
A^2 - DB^2 = N, \quad u^2 - Dv^2 = \pm 1, \quad B > 0, \quad u > 0, \quad v > 0 \quad \text{and} \quad B \leq |V|.
\]

Then

(i) \( A \geq 0 \Rightarrow U > 0 \text{ and } V > 0 \),

(ii) \( A < 0 \text{ and } N > 0 \Rightarrow U < 0 \text{ and } V < 0 \),

eg. \((-4 + \sqrt{3})(2 + \sqrt{3}) = -5 - 2\sqrt{3} \quad (u^2 - Dv^2 = 1)\);

\((-4 + \sqrt{13})(18 + 5\sqrt{13}) = -7 - 2\sqrt{13} \quad (u^2 - Dv^2 = -1)\);

(iii) \( A < 0 \text{ and } N < 0 \Rightarrow U > 0 \text{ and } V > 0 \),

eg. \((-1 + \sqrt{3})(2 + \sqrt{3}) = 1 + \sqrt{3} \quad (u^2 - Dv^2 = 1)\);

eg. \((-4 + 3\sqrt{13})(18 + 5\sqrt{13}) = 123 + 34\sqrt{13} \quad (u^2 - Dv^2 = -1)\);

(b) Let \( U + V \sqrt{D} = (A + B \sqrt{D})(u - v \sqrt{D}) \), where

\[
A^2 - DB^2 = N, \quad u^2 - Dv^2 = \pm 1, \quad B > 0, \quad u > 0, \quad v > 0 \quad \text{and} \quad B \leq |V|.
\]

Then

(i) \( A < 0 \Rightarrow U < 0 < V \),

(ii) \( A \geq 0 \text{ and } N > 0 \Rightarrow V < 0 < U \),

eg. \((4 + \sqrt{3})(2 - \sqrt{3}) = 5 - 2\sqrt{3} \quad (u^2 - Dv^2 = 1)\);

eg. \((4 + \sqrt{13})(18 - 5\sqrt{13}) = 7 - 2\sqrt{13} \quad (u^2 - Dv^2 = -1)\);

(iii) \( A \geq 0 \text{ and } N < 0 \Rightarrow U < 0 < V \),

eg. \((1 + \sqrt{3})(2 - \sqrt{3}) = -1 + \sqrt{3} \quad (u^2 - Dv^2 = 1)\);

eg. \((4 + 3\sqrt{13})(18 - 5\sqrt{13}) = -123 + 34\sqrt{13} \quad (u^2 - Dv^2 = -1)\);

Remark. (a) and (b) (i) are obvious. (b) (ii) and (iii) follow by conjugation from (a)(ii) and (iii).
Proof. We first prove (a)(ii). Assume $A < 0$.

$$U = -|A|u + BDv \quad (1)$$
$$V = -|A|v + Bu. \quad (2)$$

First assume $u^2 - Dv^2 = 1$.

Now $u > \sqrt{Dv}$ and $|A| > \sqrt{DB}$. Hence $|A|u > BDv$ and by equation (1) we have $U < 0$.

Next $|A| \leq |U|$ as $B \leq |V|$ and $A^2 = DB^2 + N, U^2 = DV^2 + N$. Hence

$$|A| \leq |A|u - BDv$$
$$BDv \leq |A|(u - 1)$$
$$BDvu \leq |A|(u - 1)u$$

However

$$|A|(u - 1)u < |A|v^2D \iff (u - 1)u < v^2D \iff 1 = u^2 - Dv^2 < u.$$ 

Hence $BDvu < |A|v^2D$ and so $Bu < |A|v$. Then equation (2) implies $V < 0$.

Now assume $u^2 - Dv^2 = -1$.

We have $u < \sqrt{Dv}$ and $\sqrt{DB} < |A|$. Hence $u\sqrt{DB} < |A|\sqrt{Dv}$ and $uB < |A|v$.

Hence from equation (2), $V < 0$.

Also $B \leq |V| = |A|v - Bu$. Hence $B(1 + u) < |A|v$.

We want to prove $|A|u > DBv$ ie. $|A| > DBv/u$.

But $|A| > B(1 + u)/v$, so it suffices to prove

$$B(1 + u)/v \geq DBv/u,$$

or $u \geq Dv^2 - u^2 = 1$.

Proof of (a)(iii). Assume $A < 0$.

First assume $u^2 - Dv^2 = 1$ and $A^2 - DB^2 = -|N|$. Then $u > \sqrt{Dv}$ and $|A| < \sqrt{DB}$. Hence $|A|v < Bu$ and $V > 0$.

Next, we have to show $BDv > |A|u$. Suppose instead that $BDv \leq |A|u$.

Now $B \leq |V|$ and (2) give $B \leq Bu - |A|v$. Hence

$$B(u - 1) \geq |A|v$$
$$B(u - 1)/v \geq |A|.$$
Hence $u(B(u - 1)/v \geq BDv$ and we deduce that $1 \geq u$, a contradiction.

Secondly, assume $u^2 - Dv^2 = -1$ and $A^2 - DB^2 = -|N|$. Now $u < \sqrt{D}v$ and $|A| < \sqrt{DB}$. Hence $u|A| < DBv$ and equation (2) gives $U > 0$.

We prove $Bu > |A|v$ by contradiction. Suppose $Bu \leq |A|v$. Then $B \leq |V| = |A|v - Bu$ and

\[
B(1 + u) \leq |A|v \\
B(1 + u)/v \leq A < \sqrt{DB} \\
(1 + u)/v < \sqrt{D}.
\]

Hence $(1 + u)^2 < Dv^2 = u^2 + 1$ and we have $2u < 0$, a contradiction.

Hence $Bu > |A|v$ and hence $V > 0$.

**Corollary** Let $x_0 + y_0\sqrt{D}$ be a fundamental solution for a class of solutions to $x^2 - Dy^2 = N$. Also let $\eta$ be the fundamental solution of the Pell’s equation $x^2 - Dy^2 = 1$ and let

\[(x_0 + y_0\sqrt{D})\eta^n = x_n + y_n\sqrt{D}.
\]

Then

(a) Suppose $N > 0$.

(i) Suppose $x_0 > 0$. Then if $n > 0$, we have $x_n > 0$ and $y_n > 0$, while if $n < 0$, we have $x_n > 0$ and $y_n < 0$.

(ii) Suppose $x_0 < 0$. Then if $n > 0$, we have $x_n < 0$ and $y_n < 0$, while if $n < 0$, we have $x_n < 0$ and $y_n > 0$.

(b) Suppose $N < 0$.

(i) Suppose $x_0 > 0$. Then if $n > 0$, we have $x_n > 0$ and $y_n > 0$, while if $n < 0$, we have $x_n < 0$ and $y_n > 0$.

(ii) Suppose $x_0 < 0$. Then if $n > 0$, we have $x_n > 0$ and $y_n > 0$, while if $n < 0$, we have $x_n < 0$ and $y_n > 0$. 

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