Hilbert's Inequalities

1. **Introduction**. In 1971, as an off–shoot of my research on the Davenport–Halberstam inequality involving well–spaced numbers, I took the numbers to be equally spaced and was led to the inequality

$$|\sum_{\substack{r,s=1\\ r\neq s}}^{R} \bar{x}_r x_s \operatorname{cosec}\frac{\pi(r-s)}{R}| \le (R-1)\sum_{r=1}^{R}|x_r|^2 \tag{1}$$

for all complex numbers $x_1,\ldots,x_R$.

If we let $n\in\mathbb{N}$ and $R\ge n$ and take $x_r=0$ for $n<r\le R$, then dividing both sides of inequality (1) by R and letting $R\to\infty$ gives Hilbert's "second" inequality

$$|\sum_{\substack{r,s=1\\ r\neq s}}^{n} \frac{\bar{x}_r x_s}{r-s}| \le \pi\sum_{r=1}^{n}|x_r|^2. \tag{2}$$

(See Hardy, Littlewood, Pólya [1].)

On reading my manuscript, Hugh Montgomery observed that a strengthening of Hilbert's inequality could be obtained:

$$|\sum_{\substack{r,s=1\\ r\neq s}}^{R} \frac{\bar{x}_r x_s}{r-s}| \le \pi(1-\frac{1}{R})\sum_{r=1}^{R}|x_r|^2. \tag{3}$$

Montgomery conjectured that if the largest eigenvalue of the Hilbert matrix is $i\mu_R$, $\mu_R>0$, then

$$\pi-\mu_R\sim\frac{c\log R}{R}.$$

He was able to obtain a weaker "order of magnitude" result.

I have not seen a reference to (3).

We remark that the "first" Hilbert inequality

$$|\sum_{r,s=1}^{R}\frac{\bar{x}_r x_s}{r+s}| \le \pi\sum_{r=1}^{R}|x_r|^2. \tag{4}$$

is much less mysterious and also follows with some care, from inequality (1). Also (see Wilf [4, pages 2–5]) if λ_R is the largest eigenvalue of the corresponding (positive definite) Hilbert matrix, then Theorem 2.2 of Wilf [4] states that

$$\lambda_R = \pi - \pi^5/2(\log R)^2 + O(\log\log R/(\log R)^3).$$

2. **A skew–circulant matrix.**

LEMMA 1. The eigenvalues of the skew–circulant matrix complex matrix

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{R-1} \\ -a_{R-1} & a_0 & \cdots & a_{R-2} \\ \vdots & \vdots & \vdots & \vdots \\ -a_1 & -a_2 & \cdots & a_0 \end{bmatrix}$$

are given by

$$\lambda_s = \sum_{r=0}^{R-1} a_r e^{\frac{(2s-1)r\pi i}{R}}, \quad s = 1, \ldots, R.$$

Proof. (Montgomery)

Regard $a_0, \ldots, a_{R-1}$ as indeterminates. Let X_s, $s = 1, \ldots, R$ be the column vector with entries

$$e^{\frac{(2s-1)r\pi i}{R}}, \quad r = 0, \ldots, R-1.$$

Then direct calculation reveals that

$$AX_s = \left(\sum_{r=0}^{R-1} a_r e^{\frac{(2s-1)r\pi i}{R}} \right) X_s.$$

LEMMA 2. Let $C = [c_{rs}]$ be the $R \times R$ matrix defined by

$$c_{rs} = \begin{cases} \operatorname{cosec} \frac{\pi(r-s)}{R} & \text{if } r \neq s \\ 0 & \text{if } r = s. \end{cases}$$

Then the eigenvalues of C are the purely imaginary numbers

$$(2s - 1 - R)i, \quad s = 1, \ldots, R,$$

or in other words, the numbers $\pm(R-1)i, \pm(R-3)i, \ldots$

PROOF. In the nomenclature of Lemma 1, $C = A$, where

$$a_0 = 0 \text{ and } a_r = -\text{cosec}\,\frac{\pi r}{R},\ r = 1, \ldots, R-1.$$

Hence the eigenvalues of C are given by

$$\begin{aligned}\lambda_s &= -\sum_{r=1} e^{\frac{(2s-1)r\pi i}{R}} \text{cosec}\,\frac{\pi r}{R} \\ &= -i\sum_{r=1}^{R-1} \frac{\sin\frac{(2s-1)\pi r}{R}}{\sin\frac{\pi r}{R}}.\end{aligned}$$

From the first of these expressions for λ_s we deduce that $\lambda_{\nu+1} - \lambda_\nu = 2i$. Also $\lambda_1 = -(R-1)i$. Consequently the theorem follows.

Noting that the eigenvalue of largest modulus of C is $i(R-1)$, we have the following result:

COROLLARY. For all complex numbers $x_1, \ldots, x_R$,

$$|\sum_{r\neq s}^{R} \bar{x}_r x_s \text{cosec}\,\frac{\pi(r-s)}{R}| \leq (R-1)\sum_{r=1}^{R} |x_r|^2.$$

PROOF. A skew–symmetric matrix C is a normal matrix and is hence unitarily similar to a diagonal matrix. We then argue as in the proof of Theorem 12.6.5, Mirsky [2, page 388].

3. **An improvement to Hilbert's inequality**.

The next result is due to Schur (see Satz 5, Mirsky [3, page 11].)

LEMMA 3. Let $C = [c_{rs}]$ and $D = [d_{rs}]$ be $R \times R$ matrices with D positive definite Hermitian. Then if $\mu = \max_r d_{rr}$ and ν is a positive number such that the inequality

$$|\sum_{r=1}^{R}\sum_{s=1}^{R} \bar{x}_r x_s c_{rs}| \leq \nu \sum_{r=1}^{R} |x_r|^2$$

holds for all complex numbers $x_1, \ldots, x_R$, then the inequality

$$|\sum_{r=1}^{R}\sum_{s=1}^{R} \bar{x}_r x_s c_{rs} d_{rs}| \leq \mu\nu \sum_{r=1}^{R} |x_r|^2$$

holds for all complex numbers $x_1, \ldots, x_R$.

We are now able to derive the improvement in Hilbert's second inequality, as pointed out by Montgomery:

THEOREM. The inequality

$$|\sum_{\substack{r,s=1 \\ r \neq s}}^{R} \frac{\bar{x}_r x_s}{r-s}| \leq \pi(1-\frac{1}{R}) \sum_{r=1}^{R} |x_r|^2 \tag{5}$$

for all complex numbers $x_1, \ldots, x_R$.

PROOF. We have

$$\sum_{\substack{r,s=1 \\ r \neq s}}^{R} \frac{\bar{x}_r x_s}{r-s} = \sum_{r=1}^{R} \sum_{s=1}^{R} \bar{x}_r x_s c_{rs} d_{rs}, \tag{6}$$

where

$$\begin{aligned} c_{rs} &= \begin{cases} \operatorname{cosec} \frac{\pi(r-s)}{R} & \text{if } r \neq s \\ 0 & \text{if } r = s. \end{cases} \\ d_{rs} &= \begin{cases} \frac{1}{r-s} \sin \frac{\pi(r-s)}{R} & \text{if } r \neq s \\ \frac{\pi}{R} & \text{if } r = s. \end{cases} \end{aligned}$$

It is easy to prove that D is positive definite. For

$$\begin{aligned} \frac{R}{\pi} \sum_r \sum_s \bar{x}_r x_s d_{rs} &= \frac{R}{\pi} \sum_{r \neq s} \bar{x}_r x_s d_{rs} + \sum_r |x_r|^2 \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_r \sum_s \bar{x}_r x_s e^{\frac{2\pi(s-r)ix}{R}} dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} |\sum_r x_r e^{\frac{2\pi rix}{R}}|^2 dx. \end{aligned}$$

Since $d_{rr} = \pi/R$, we have $\mu = \pi/R$. Also by the Corollary, we may take $\nu = R-1$. Consequently (5) follows from (6) and Lemma 3.

Dr. Graham Jameson has supplied another proof of the Theorem, which does not use Lemma 3.

Let

$$b_{r,s} = \begin{cases} 1/(r-s) & \text{if } r \neq s, \\ 0 & \text{if } r = s, \end{cases}$$

and let

$$S = \sum_{r=1}^{R} \sum_{s=1}^{R} b_{r,s} \overline{x_r} x_s.$$

With the notation of the paper, $b_{r,s} = c_{r,s} d_{r,s}$.

Write (as usual) $e(t) = e^{2\pi i t}$. Note first that

$$\int_{-1/2}^{1/2} e(\lambda t)\, dt = \frac{\sin \pi\lambda}{\pi\lambda}$$

for $\lambda \neq 0$. Hence

$$\begin{aligned} d_{r,s} &= \frac{\pi}{R} \int_{-1/2}^{1/2} e\left(\frac{(r-s)t}{R}\right) dt \\ &= \frac{\pi}{R} \int_{-1/2}^{1/2} g_r(t) \overline{g_s(t)}\, dt, \end{aligned}$$

where $g_r(t) = e(rt/R)$ (also when $r = s$). So

$$S = \frac{\pi}{R} \int_{-1/2}^{1/2} \sum_{r=1}^{R} \sum_{s=1}^{R} c_{r,s}\, g_r(t) \overline{g_s(t)}\, \overline{x}_r x_s\, dt.$$

By (1), applied to the scalars $\overline{g_r(t)} x_r$, we have

$$\left| \sum_{r=1}^{R} \sum_{s=1}^{R} c_{r,s}\, g_r(t) \overline{x}_r\, \overline{g_s(t)} x_s \right| \leq (R-1) \|x\|^2.$$

Hence

$$|S| \leq \frac{\pi}{R}(R-1)\|x\|^2 = \pi \left(1 - \frac{1}{R}\right) \|x\|^2.$$

References

[1] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press 1959.

[2] L. Mirsky, *An introduction to linear algebra*, Oxford University Press 1961.

[3] I. Schur, *Bemerkungungen zur Theorie der beschränkten Bilinearformen mit endlich vielen Verändlichen*, J. reine angew. Math., 140 (1911), 1–28.

[4] H.S. Wilf, *Finite sections of some classical inequalities*, Ergebnisse der Mathematik, Band 52, Springer 1970.