An inequality for the determinant of a positive definite Hermitian matrix

The following result was obtained by Professor Grainger Morris, formerly of the University of New England. Grainger asked for a simple proof and I came up with the following on 18th November 1996.

THEOREM If $H = A + iB$ is a positive–definite Hermitian matrix, $A, B$ real, then $\det H \leq \det A$.

PROOF Let $H = A + iB$ and $D = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$. Then

1. $H$ is Hermitian if and only if $A$ is symmetric and $B$ is skew–symmetric.
2. $H$ is Hermitian if and only if $D$ is symmetric.
3. $H$ is Hermitian positive–definite if and only if $D$ is symmetric positive–definite.
4. $\det D = (\det H)^2$. In fact $ch_D(x) = (ch_H(x))^2$. (Use elementary row and column operations.)
5. If $D$ is symmetric positive definite, then by Fischer’s inequality (see L. Mirsky, *An Introduction to Linear Algebra*, OUP 1961, Theorem 13.5.5, page 420), we have $\det D \leq (\det A)^2$.

Then the desired inequality $\det H \leq \det A$ follows from (3) and (4).