SOLVING THE GENERAL QUADRATIC DIOPHANTINE EQUATION $ax^2 + bxy + cy^2 + dx + ey + f = 0$

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1. Introduction

This note originates from studying the paper [11, pp. 38–40] where a new method of solving the general quadratic diophantine equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{1}$$

is given in the case where (1) represents an hyperbola. This was an improvement on a classical method of Lagrange mentioned in [11, p. 39].

In our note, we give a variation of the method of [11] due to John Robertson, which uses a transformation of variables where the centre of the hyperbola becomes the origin.

The rest of the note is a standard treatment of the special cases that correspond to an ellipse, parabola, or two straight lines, possibly coincident.

The underlying algorithm has been coded as BCMath program [3].

An earlier computer program for solving (1) due to Dario Alpern is available at [4].

2. The cases

Let $D = b^2 - 4ac$. We assume not all of a, b, c are zero.

Case 1. $D = 0$. We use completion of the square, as in Hua's book [8, p. 278]. We can assume $a \neq 0$ (by interchanging x and y, as one of a and c is nonzero) and multiply (1) by $4a$ to get an equivalent equation:

$$(2ax + by)^2 + 4adx + 4aey + 4af = 0. \tag{2}$$

Let $t = 2ax + by$. Then (2) becomes

$$(t + d)^2 = uy + v. \tag{3}$$

where $u = 2(bd - 2ae)$ and $v = d^2 - 4af$.

(i) Assume $u = 0$. Then (3) becomes $(t + d)^2 = v$. Let $h = \gcd(2a, b)$.

If $v = 0$, then equation (3) becomes $2ax + by + d = 0$ and we have a line of integer solutions $(2a/h)x + (b/h)y = d/h$ for (1) if h divides d, but no integer solution if h does not divide d.

If $v < 0$ or $v > 0$ and v is nonsquare, there is no integer solution of (1).

Next assume $v = g^2, g > 0$. Then $t + d = \pm g$, i.e.,

$$2ax + by = \pm g - d$$

and we have the following possibilities:

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(a) If h divides $g-d$, we have a line of integer solutions of (1):

$$(2a/h)x + (b/h)y = (g-d)/h.$$

(b) If h divides $g+d$, we have a line of integer solutions of (1):

$$(2a/h)x + (b/h)y = (-g-d)/h.$$

(c) Neither $g-d$ nor $g+d$ is divisible by h. Then there is no integer solution of (1).

(ii) Assume $u \neq 0$. Then (3) gives rise to the congruence

$$T^2 \equiv v \pmod{|u|}, \tag{4}$$

where $T = t + d$ and $t = ax + by$.

If there are no solutions of (4), then there are no integer solutions of (1). So let $T_1, \ldots, T_c$ be the solutions in the range $-|u|/2 < T_i \leq |u|/2$. Then

$$t = T_i - d + uk, \ k \text{ arbitrary}.$$

We now process each T_i. Equation (3) gives

$$y = r + sk + uk^2, \tag{5}$$

where $r = (T_i^2 - v)/u$ and $s = 2T_i$. We have to choose k such that x is integral in the equation

$$2ax + by = t = T_i - d + ku,$$

or equivalently

$$2ax = T_i - d - br + (u - bs)k - buk^2. \tag{6}$$

Fortunately $-bu/2a$ is an integer. For $bu = 2b(bd - 2ae) = 2b^2d - 4bae$ and since $b^2 = 4ac$, we see that $-bu/2a = 2be - 4cd$, an integer. Hence we have to solve the congruence

$$(u - bs)k \equiv d + br - T_i \pmod{2a}.$$

If this is not soluble, we pass on to T_{i+1}. Otherwise let the solution have the form

$$k = x' + zw, \text{ where } z \geq 1, 0 \leq x' < z \text{ and } w \text{ is arbitrary.}$$

Then equation (5) gives

$$\begin{aligned} y &= r + s(x' + zw) + u(x' + zw)^2 \\ &= r + sx' + ux'^2 + (sz + 2ux'z)w + uz^2w^2. \end{aligned}$$

Also, with $j = u - bs, K = d + br - T_i$ and $t = 2be - 4cd$, equation (6) gives

$$x = (x'j - K)/2a + tx'^2 + (zj/2a + 2tx'z)w + t^2z^2w^2.$$

Note that t is nonzero, as $t = 0$ implies $u = 0$. The solutions (x, y) lie on a parabola.

Case 2. $D \neq 0$. We multiply (1) by D^2 and translate the origin to (α, β), where

$$\alpha = 2cd - be, \beta = 2ae - bd,$$

using the transformation of Legendre ([9, p. 105])

$$Dx = X + \alpha, Dy = Y + \beta,$$

to get the equation

$$aX^2 + bXY + cY^2 = k, \tag{7}$$

where

$$k = -D(ae^2 - bed + cd^2 + fD).$$

Let $t = \gcd(a, b, c)$. If t does not divide k, then (1) has no integer solution. Otherwise we replace (a, b, c, k) by $(a/t, b/t, c/t, k/t)$ in (7).

(a) Assume $k = 0$ and D not a square. Then the only integer solution of (7) is $(X, Y) = (0, 0)$ and so $Dx = \alpha$ and $Dy = \beta$. Hence if $D|\alpha$ and $D|\beta$, we have the unique solution $(x, y) = (\alpha/D, \beta/D)$, whereas if D does not divide α or D does not divide β, there is no integer solution of (1).

(b) Assume $D < 0$ and $k \neq 0$. Then (1) describes an ellipse. We use the recent algorithm of Matthews ([10]) to find all integer solutions (X_i, Y_i) of $aX^2 + bXY + cY^2 = k$. If D divides $X_i + \alpha$ and D divides $Y_i + \beta$, we get a corrresponding integer solution of (1):

$$(x, y) = ((X_i + \alpha)/D, (Y_i + \beta)/D).$$

(c) Assume $D = g^2, g > 0$.

(i) Assume $a \neq 0$. Then on multiplying by $4a$, equation (7) becomes

$$(2aX + (b + g)Y)(2aX + (b - g)Y) = 4ak.$$

Let $g_1 = \gcd(2a, b + g), g_2 = \gcd(2a, b - g)$.
If g_1g_2 does not divide $4ak$, then equation (1) has no integer solution.
If g_1g_2 divides $4ak$, we now have to solve

$$\left(\frac{2a}{g_1}X + \frac{(b+g)}{g_1}Y\right)\left(\frac{2a}{g_2}X + \frac{(b-g)}{g_2}Y\right) = \frac{4ak}{g_1g_2}.$$

We consider the cases $k = 0$ and $k \neq 0$ separately:
First the case $k = 0$. We get two equations $2aX + (b \pm g)Y = 0$. Using $Dx = X + \alpha$ and $Dy = Y + \beta$, these in turn give two equations

$$2aDx + (b + g)Dy = 2a\alpha + (b + g)\beta \tag{8}$$

$$2aDx + (b - g)Dy = 2a\alpha + (b - g)\beta. \tag{9}$$

If Dg_1 does not divide $2a\alpha + (b+g)\beta$, then equation (8) does not lead to a solution for (x, y).
If Dg_1 divides $2a\alpha + (b+g)\beta$, then we get the line of integer solutions:

$$(2a/g_1)x + ((b + g)/g_1)y = (2a\alpha + (b + g)\beta)/Dg_1.$$

Similarly for (9).
Secondly, consider the case $k \neq 0$. We are dealing with an equation of the form

$$(A_1X + B_1Y)(A_2X + B_2Y) = 4ak/(g_1g_2),$$

where $A_1 = 2a/g_1, A_2 = 2a/g_2, B_1 = (b + g)/g_1, B_2 = (b - g)/g_2$.
We have to examine all divisors d_i of $4ak/(g_1g_2)$ and test for integer solutions (X, Y) of

$$A_1X + B_1Y = d_i \tag{10}$$

$$A_2X + B_2Y = 4ak/(g_1g_2d_i). \tag{11}$$

If there are no integer solutions of the system of equations (10) and (11) for any i, then there are no integer solutions of (1).
However if there is an i such that (10) and (11) have integer solutions (X, Y), then each such solution, we have to check if D divides $X + \alpha$ and $Y + \beta$, in which case we get an integer solution of (1):

$$(x, y) = ((X + \alpha)/D, (Y + \beta)/D).$$

(ii) Assume $a = 0$. Then (7) becomes $Y(bX + cY) = k$. Again we consider the cases $k \neq 0$ and $k = 0$ separately. Let $h = \gcd(b, c)$.
First assume $k \neq 0$. If h does not divide k, then (1) has no integer solutions.
If h divides k, we get the equation

$$Y((b/h)X + (c/h)Y) = k/h.$$

We then have to examine all divisors d_i of k/h and solve the system

$$\begin{aligned} Y &= d_i \\ (b/h)X + (c/h)Y &= k/(hd_i) \end{aligned}$$

in integers.
Secondly, assume $k = 0$. Then (3) becomes $Y(bX + cY) = 0$, i.e.,

$$(Dy + \beta)(bDx + cDy + b\alpha + c\beta) = 0.$$

.
If D divides β, we get one family of integer solutions of (1), namely $y = \beta/D$, with x arbitrary. Let $g' = \gcd(b, c)$ and $t = b\alpha + c\beta$.
If $g'D$ does not divide t, there are no integer solutions of (1).
If $g'D$ divides t, we get a line of integer solutions of (1):

$$(b/g')x + (c/g')y + t/Dg' = 0.$$

(d) Assume $D > 0$ and nonsquare and $k \neq 0$. Then (1) describes an hyperbola. We assume that for the original a, b, c, we have $\gcd(a, b, c) = 1$. We distinguish between D odd and even.

(i) D odd. Let (ϕ, ψ) be the least positive solution of $\phi^2 - D\psi^2 = 4$. We then find representatives (X_i, Y_i) for the solution classes of equation (7). The general solution for each class is given by

$$\begin{aligned} X &= \gamma F + \delta G \\ Y &= \epsilon F + \zeta G, \end{aligned}$$

where

$$\gamma = X_i, \epsilon = Y_i, \delta = -(2c\epsilon + b\gamma), \zeta = b\epsilon + 2a\gamma$$

and $F + G\sqrt{D} = \pm(\phi/2 + (\psi/2)\sqrt{D})^n, n \in \mathbb{Z}$.
By a result of John Robertson, it is enough to test $\gamma F + \delta G + \alpha$ and $\epsilon F + \zeta G + \beta$ for divisibility by D when $(F, G) = (1, 0), (-1, 0), (\phi/2, \psi/2)$ and $(-\phi/2, -\psi/2)$.

When divisibility is successful, we get a corresponding solution family (x, y) of the form

$$Dx = \gamma F + \delta G + \alpha$$
$$Dy = \epsilon F + \zeta G + \beta,$$

where $F + G\sqrt{D} = s(\phi/2 + (\psi/2)\sqrt{D})^{2m+t}, m \in \mathbb{Z}$, where $s = \pm 1$ and $t = 0$ or 1.

(ii) D even. Let (ϕ, ψ) be the least positive solution of $\phi^2 - D\psi^2 = 1$. We then find representatives (X_i, Y_i) for the solution classes of equation (7). The general solution for each class is given by

$$X = \gamma F + \delta G$$
$$Y = \epsilon F + \zeta G,$$

where

$$\gamma = X_i, \epsilon = Y_i, \delta = -(c\epsilon + (b/2)\gamma), \zeta = (b/2)\epsilon + a\gamma$$

and $F + G\sqrt{D} = \pm(\phi + \psi\sqrt{D})^n, n \in \mathbb{Z}$.
Then by a result of John Robertson, it is enough to test $\gamma F + \delta G + \alpha$ and $\epsilon F + \zeta G + \beta$ for divisibility by D when $(F, G) = (1, 0), (-1, 0), (\phi, \psi)$ and $(-\phi, -\psi)$.
When divisibility is successful, we get a corresponding solution family (x, y) of the form

$$Dx = \gamma F + \delta G + \alpha$$
$$Dy = \epsilon F + \zeta G + \beta,$$

where $F + G\sqrt{D} = s(\phi + \psi\sqrt{D})^{2m+t}, m \in \mathbb{Z}$, where $s = \pm 1$ and $t = 0$ or 1.

(e) Lastly we assume $D > 0$, nonsquare and $k \neq 0$ and that for the original a, b, c, we have $\gcd(a, b, c) = g > 1$. We return to the original equation (1) and make the transformation $gx = x', gy = y'$ in equation (1) and consider instead the equation

$$a'x'^2 + b'x'y' + c'y'^2 + dx' + ey' + fg = 0, \tag{12}$$

where $a' = a/g, b' = b/g, c' = c/g$ and $D' = b'^2 - 4a'c'$. Then $\gcd(a', b', c') = 1$ and we make the transformation $D'x' = X + \alpha', D'y' = Y + \beta'$ to get an equation

$$a'X^2 + b'XY + c'Y^2 = k'. \tag{13}$$

Then, as in the previous case, we find solution classes (x', y') for (12) of the form

$$D'gx = D'x' = \gamma F_m + \delta G_m + \alpha' = X_m$$
$$D'gy = D'y' = \epsilon F_m + \zeta G_m + \beta' = Y_m,$$

where $\phi^2 - D'\psi^2 = 4$ and $F_m + G_m\sqrt{D'} = s(\phi/2 + (\psi/2)\sqrt{D'})^{2m+t}, m \in \mathbb{Z}$, where $s = \pm 1$ and $t = 0$ or 1, when D' is odd, and

$$D'x' = \gamma F_m + \delta G_m + \alpha' = X_m$$
$$D'y' = \epsilon F_m + \zeta G_m + \beta' = Y_m,$$

where $\phi^2 - D'\psi^2 = 1$ and $F_m + G_m\sqrt{D'} = s(\phi + \psi\sqrt{D'})^{2m+t}, m \in \mathbb{Z}$, where $s = \pm 1$ and $t = 0$ or 1, when D' is even.

Then X_n and Y_n satisfy recurrence relations of the form

$$X_{n+1} = u_1 X_n - X_{n-1} + w_1$$
$$Y_{n+1} = u_2 Y_n - Y_{n-1} + w_2.$$

(See the Appendix for definitions of u_1, u_2, w_1, w_2.)

We determine the period (mod $D'g$) of the sequence (X_n, Y_n) and test each (X_n, Y_n) in the first period for divisibility by $D'g$. Then each case of divisibility where $X_{n_i} = gD'x_{n_i}$ and $Y_{n_i} = gD'y_{n_i}$, will yield a solution family $(x_n, y_n), n = 2(D'gm + n_i) + t$ of (1). There may be several such solutions found in such a period.

3. Examples

(1) $x^2 - 15y^2 = 61$. [8, p. 285]. Ans. $x + y\sqrt{15} = \pm(11 \pm 2\sqrt{15})(4 + \sqrt{15})^n$.

(2) $3x^2 - 8xy + 7y^2 - 4x + 2y - 109 = 0$. Exercise (a), [8, p. 286].

Ans. $D = -20, (2, 5), (2, -3), (14, 9), (-10, -7)$.

(3) $3xy + 2y^2 - 4x - 3y - 12 = 0$. Exercise (b), [8, p. 286].

Ans. $D = 9$,

$$(5, 2), (-3, 6), (-1, 4), (-13, 20), (-13, 1),$$
$$(-1, -1), (-3, 0), (5, -8), (2, -4), (24, -36).$$

(4) $9x^2 - 12xy + 4y^2 + 3x + 2y - 12 = 0$. Exercise (c), [8, p. 286].

Ans. $D = 0$. Four families:

$$x = 2 - 2t - 24t^2, x = 0 + 14t - 24t^2, x = 1 + 10t - 24t^2, x = -5 + 26t - 24t^2,$$
$$y = 3 + 3t - 36t^2, y = -2 + 27t - 36t^2, y = 0 + 21t - 36t^2, y = -11 + 45t - 36t^2.$$

(5) $x^2 - 8xy - 17y^2 + 72y - 75 = 0$. Exercise (d), [8, p. 286].

Ans. $D = 132$. Two families: With $F + G\sqrt{33} = -(23 + 4\sqrt{33})^n$ and

$$11x = 70F + 297G + 48, \quad 11x = -62F - 231G + 48,$$
$$11y = F + 66G + 12, \quad 11y = F - 66G + 12,$$

(6) $x^2 + 8xy + y^2 + 2x - 4y + 1 = 0$. Art. 221, [7, p. 220–221].

Ans. $D = 60$. Two families:

$$5x = -8F + 30G + 3$$
$$5y = 2F - 2,$$

where $F + G\sqrt{15} = (4 + \sqrt{15})^{2m}$ or $-(4 + \sqrt{15})^{2m+1}$. We remark that equivalently F and G satisfy $F^2 - 15G^2 = 1$ and $F \equiv 1 \pmod 5$. This then relates more readily to the solution given by Gauss, namely

$$5x = 2t + 3$$
$$5y = -8t + 30u - 2,$$

where $t^2 - 15u^2 = 1$ and $t \equiv 1 \pmod 5$. These solutions correspond under the transformation

$$t = -4F + 15G, u = -F + 4G.$$

(7) $x^2+2xy+y^2+x+y-6=0$.

Ans. $D=0$. $x+y=2$ and $x+y=-3$.

(8) $2x^2+7xy+6y^2+5x+7y-27=0$.

Ans. $D=1$. Sixteen solutions:

$$(56,-29),(18,-9),(29,-15),(1,1),(11,-5),(-21,15),(-7,6),(-59,40),$$
$$(-34,15),(4,-5),(-7,1),(21,-15),(11,-9),(43,-29),(29,-20),(81,-54).$$

(9) $81x^2+78xy+22y^2=225$. Exercise III.3, [6, p. 115].

Ans. $D=-1044$. Two solutions: $(3,-3),(-3,3)$.

(10) $2x^2+3xy+5y^2=3352$. Exercise III.4, [6, p. 115].

Ans. $D=-31$. Six solutions: $(31,10),(-31,-10),(-17,-19),(17,19),(46,-10),(-46,10)$. Note. Faisant does not list four of these solutions.

(11) $3x^2-22xy+25y^2=81$. Exercise III.6, [6, p. 115].

Ans. $D=184$. Five families of solutions:

$$x=-658F-4463G,\quad x=-2F-47G,\quad x=-156F+1059G$$
$$y=-111F-753G,\quad y=F-17G,\quad y=-111F+753G,$$

$$x=54F+369G,\quad x=-12F+93G,$$
$$y=9F+63G,\quad y=-9F+63G,$$

with $F+G\sqrt{46}=\pm(24335+1794\sqrt{46})^m$. Note. Faisant has an incorrect answer $(-111,-156)$ instead of $(-156,-111)$.

(12) $2x^2+8xy-y^2-4x+10y-7=0$. Ans. $(-1,1)$.

(13) $41x^2+25xy-125y^2-10x+9y-7=0$. An example of John Robertson, April 13, 2015.

Ans. $D=21125$. Two families of solutions:
representatives $(3,2)$ and $(-1224927844935779,-834638485983846)$,
with corresponding general solutions given by

$$65x=188F+27040G+7$$
$$1625y=3174F+484750G+76,$$

where respectively $F+G\sqrt{21225}=\epsilon^{2m}$ or $-\epsilon^{2m+1}$, with
$\epsilon=425730551631123/2+(2929115241679/2)\sqrt{21125}=(843/2+(29/2)\sqrt{845})^5$.

(14) $3x^2+14xy+6y^2-17x-23y-505=0$. An example of Dario Alpern [5].

Ans. $D=124$. Eight families of solutions:

(i) $(610,-1275)$ with corresponding general solution given by

$$62x=-7809F+43617G+59$$
$$62y=1841F-10540G+50,$$

where $F+G\sqrt{31}=(1520+273\sqrt{31})^{2m+1}$;

(ii) $(-31,7)$ with corresponding general solution given by

$$62x=-1981F+11563G+59$$
$$62y=384F-3255G+50,$$

where $F+G\sqrt{31}=(1520+273\sqrt{31})^{2m}$;

(iii) $(-1291,2707)$ with corresponding general solution given by

$$62x=-3661F+20677G+59$$
$$62y=825F-5208G+50,$$

where $F+G\sqrt{31}=-(1520+273\sqrt{31})^{2m+1}$;

(iv) $(70, -15)$ with corresponding general solution given by

$$62x = -4281F + 24087G + 59$$
$$62y = 980F - 5983G + 50,$$

where $F + G\sqrt{31} = -(1520 + 273\sqrt{31})^{2m}$;

(v) $(-125, 265)$ with corresponding general solution given by

$$62x = -7809F - 43617G + 59$$
$$62y = 16380F + 91233G + 50,$$

where $F + G\sqrt{31} = (1520 + 273\sqrt{31})^{2m}$;

(vi) $(4, 7)$ with corresponding general solution given by

$$62x = -189F + 3627G + 59$$
$$62y = -384F - 3255G + 50,$$

where $F + G\sqrt{31} = -(1520 + 273\sqrt{31})^{2m}$;

(vii) $(211015, -441995)$ with corresponding general solution given by

$$62x = -4281F - 24087G + 59$$
$$62y = 9009F + 50220G + 50,$$

where $F + G\sqrt{31} = -(1520 + 273\sqrt{31})^{2m+1}$;

(viii) $(20605, -43157)$ with corresponding general solution given by

$$62x = -189F - 3627G + 59$$
$$62y = 825F + 5208G + 50,$$

where $F + G\sqrt{31} = -(1520 + 273\sqrt{31})^{2m+1}$;

(15) $6x^2 + 24xy + 12y^2 + x - y + 7 = 0.$

Ans. $D = 288$. Two families of solutions:

(i) $(2, -3)$ with corresponding general solution given by

$$12x = -16F + 13G + 2$$
$$24y = 19F + 6G - 3,$$

where $F + G\sqrt{2} = -(3 + 2\sqrt{2})^{4m+1}$;

(ii) $(13, -22)$ with corresponding general solution given by

$$12x = -14F + 7G + 2$$
$$24y = 21F + 14G - 3,$$

where $F + G\sqrt{2} = -(3 + 2\sqrt{2})^{4m+2}$;

(16) $6x^2 + 30xy + 18y^2 + x - y - 11 = 0.$

Ans. $D = 468$. Two families of solutions:

(i) $(2, -3)$ with corresponding general solution given by

$$78x = -158F + 676G + 11$$
$$78y = 19F - 221G - 7,$$

where $F + G\sqrt{13} = (11/2 + (3/2)\sqrt{13})^{6m+1}$;

(ii) $(7, -10)$ with corresponding general solution given by

$$78x = 2F - 364G + 11$$
$$78y = 59F + 299G - 7,$$

where $F + G\sqrt{13} = -(11/2 + (3/2)\sqrt{13})^{6m+1}$;

4. Appendix

Let $D = b^2 - 4ac$, with $\Delta = D/4$ if D is even.
Suppose D is odd, let

$$X_m = \gamma F_m + \delta G_m + \alpha',$$
$$Y_m = \epsilon F_m + \zeta G_m + \beta',$$

where $\phi^2 - D\psi^2 = 4$ and $F_m + G_m\sqrt{D} = s(\phi/2 + (\psi/2)\sqrt{D})^{2m+t}, m \in \mathbb{Z}$, where $s = \pm 1$ and $t = 0$ or 1.

If D is even, let

$$X_m = \gamma F_m + \delta G_m + \alpha',$$
$$Y_m = \epsilon F_m + \zeta G_m + \beta',$$

where $\phi^2 - \Delta\psi^2 = 1$ and $F_m + G_m\sqrt{\Delta} = s(\phi + \psi\sqrt{\Delta})^{2m+t}, m \in \mathbb{Z}$, where $s = \pm 1$ and $t = 0$ or 1.

Then X_m and Y_m satisfy recurrence relations of the form

$$X_{n+1} = u_1 X_n - X_{n-1} + w_1,$$
$$Y_{n+1} = u_2 Y_n - Y_{n-1} + w_2,$$

where u_1, u_2, w_1, w_2 are defined as follows:

(a) Suppose D is even. Let

$$f = s, g = 0 \text{ if } t = 0,$$
$$f = \phi, g = \psi \text{ if } t = 1.$$

Then

$$X_0 = \gamma f + \delta g + \alpha',$$
$$Y_0 = \epsilon f + \zeta g + \beta'.$$

Let

$$f + g\sqrt{\Delta} = \begin{cases} s(\phi + \psi\sqrt{\Delta})^2 & \text{if } t = 0, \\ s(\phi + \psi\sqrt{\Delta})^3 & \text{if } t = 1. \end{cases}$$

Then

$$X_1 = \gamma f + \delta g + \alpha',$$
$$Y_1 = \epsilon f + \zeta g + \beta'.$$

Let $\phi_2 + \psi_2 = (\phi + \psi\sqrt{\Delta})^2$ and define α_1 and β_1 by

$$\alpha_1 = 2\alpha' - 2\phi_2\alpha',$$
$$\beta_1 = 2\beta' - 2\phi_2\beta'.$$

Then

$$X_{n+1} = 2\phi_2 X_n + \alpha_1 - X_{n-1},$$
$$Y_{n+1} = 2\phi_2 Y_n + \beta' - Y_{n-1}.$$

(b) D odd. Let

$$f = s, g = 0 \text{ if } t = 0,$$
$$f = \phi, g = \psi \text{ if } t = 1.$$

Then

$$X_0 = (\gamma f + \delta g)/2 + \alpha',$$
$$Y_0 = (\epsilon f + \zeta g)/2 + \beta'.$$

Let

$$f + g\sqrt{\Delta} = \begin{cases} s(\phi + \psi\sqrt{\Delta})^2 & \text{if } t = 0, \\ s(\phi + \psi\sqrt{\Delta})^3 & \text{if } t = 1. \end{cases}$$

Then

$$X_1 = (\gamma f + \delta g)/8 + \alpha',$$
$$Y_1 = (\epsilon f + \zeta g)/8 + \beta'.$$

Let $\phi_2 + \psi_2 = ((\phi + \psi\sqrt{\Delta})^2)/2$ and define α_1 and β_1 by

$$\alpha_1 = 2\alpha' - \phi_2\alpha',$$
$$\beta_1 = 2\beta' - \phi_2\beta'.$$

Then

$$X_{n+1} = \phi_2 X_n + \alpha_1 - X_{n-1},$$
$$Y_{n+1} = \phi_2 Y_n + \beta' - Y_{n-1}.$$

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