On a diophantine equation of Andrej Dujella

Keith Matthews

Let $k \geq 2$, $k \in \mathbb{N}$. Then the diophantine equation

$$x^2 - (k^2 + 1)y^2 = k^2$$

has at most one positive solution $(x, y)$ with $y < k - 1$. We call such a solution an *exceptional* solution.

Example. $k = 8$ is the first $k$ possessing an exceptional solution, namely $(x, y) = (18, 2)$.

We have verified the conjecture for $k \leq 2^{50}$. 
The conjecture has been proved in the following cases:

Filipin, Fujita and Mignotte:

(a) \( k^2 + 1 = p^n \) or \( 2p^n \), \( p \) an odd prime: no exceptional solutions.

(b) \( k = p^{2i} \) or \( p^{2i+1} \) or \( 2p^{2i+1} \), \( p \) an odd prime: no exceptional solutions.

(c) \( k = 2p^{2i} \), \( p \) an odd prime: the exceptional solution is \( (2p^{3i} + p^i, p^i) \).

Matthews and Robertson: \( k^2 + 1 = p^m q^n \) or \( 2p^m q^n \), \( m, n \geq 1 \), \( p \) and \( q \) distinct odd primes.
The unicity conjecture implies the $D(-1)$ 4–tuples conjecture (Dujella)

Assume the unicity conjecture and let $a, b, c, d$ be a $D(-1)$-quadruple with $0 < a < b < c < d$. Then $a = 1$ by Dujella-Fuchs (J. London Math. Soc. 2005) and hence

$$b = r^2 + 1, \quad c = s^2 + 1, \quad d = t^2 + 1.$$ 

Now consider the equation $(y^2 + 1)(t^2 + 1) = x^2 + 1$, i.e.,

$$x^2 - (t^2 + 1)y^2 = t^2.$$ 

By the conjecture, this diophantine equation has at most one solution with $0 < y < t - 1$.

But by assumption, it has at least two solutions with $0 < y < t$, namely, $y = r$ and $y = s$, and hence we must have $s = t - 1$.

However this contradicts a *gap* property (Dujella-Fuchs, Lemma 9) which implies that $d > c^2$, because the inequality

$$d = t^2 + 1 > c^2 = ((t - 1)^2 + 1)^2$$

does not hold for any $t > 2$. 
Type 1 and Type 2 solutions

Dujella’s equation can be written as

\[ x^2 - y^2 = (y^2 + 1)k^2. \]

We divide the exceptional solutions into two classes:

The Type 1 solutions are those for which \( y^2 + 1 \) divides \( x + y \) or \( x - y \), while Type 2 solutions are the remaining ones.

In the range \( k \leq 2^{50} \), there are 23,862,782 Type 1 and 73,034 Type 2 exceptional solutions.
Characterisation of Type 1 solutions

Proposition. There is a 1–1 correspondence between the Type 1 solutions \((x, y)\), with \(x \equiv \epsilon y \pmod{y^2 + 1}\), \(\epsilon = \pm 1\) and the integer pairs \((r, s)\) which satisfy \(1 < r < s\) and

\[
    r^2 + s^2 = k^2 + 1
\]

\[
    s \equiv \epsilon \pmod{r},
\]

namely

\[
    r = \frac{x - \epsilon y}{y^2 + 1}, \quad s = \frac{xy + \epsilon}{y^2 + 1},
\]

where we take \(\epsilon = 1\) if \(y = 1\).

Example. \(k = 8\), \((x, y) = (18, 2)\), \(\epsilon = -1\), \((r, s) = (4, 7)\).
Example 1: Type 1(a) exceptional solution

These are the \((k_n, x_n, y)\), where

\[ x_n + k_n \sqrt{D} = y(R + S \sqrt{D})^n, \quad n \geq 1, \]

and \(R = 2y^2 + 1, \ S = 2y, \ D = y^2 + 1\) and \(y \geq 2\).

Here

\[ x_n \equiv (-1)^n y \pmod{y^2 + 1} \]

and \(y\) divides \(x_n\).
Example 2: Type 1(b) exceptional solution

These are the \((k_n, x_n, y)\), where

\[ x_n + k_n \sqrt{D} = (y^2 + \epsilon y + 1 + (y + \epsilon)\sqrt{D})(R + S\sqrt{D})^n, \quad n \geq 1, \]

and where \(y \geq 1\) if \(\epsilon = 1\) and \(y \geq 2\) if \(\epsilon = -1\).

Here

\[ x_n \equiv (-1)^n \epsilon y \pmod{y^2 + 1}, \]

and \(\gcd(x_n, y) = 1\).
Types 1(a) and 1(b) give all Type 1 solutions

Theorem. If $(k, x, y)$ is a Type 1 solution, then

(i) either (a) $y$ divides $x$ and $y > 1$, or (b) $\gcd(x, y) = 1$.

(ii) $(k, x, y)$ is a Type 1(a) solution in case (a) and a Type 1(b) solution in case (b).
Producing exceptional solutions

The following three functions each create an exceptional solution \((K_i, X_i, Y_i)\) from an exceptional solution \((k, x, y)\):

(i) \(g_+(k, x, y) = (K_1, X_1, Y_1), Y_1 = k,\)

(ii) \(g_-(k, x, y) = g_+(k, x, -y) = (K_2, X_2, Y_2), Y_2 = k,\)

(iii) \(g_0(k, x, y) = g_+(y, x, k) = (K_3, X_3, Y_3), Y_3 = y,\)

where

\[
X_1 + K_1 \sqrt{k^2 + 1} = (x + y \sqrt{k^2 + 1})(2k^2 + 1 + 2k \sqrt{k^2 + 1})
\]
\[
X_2 + K_2 \sqrt{k^2 + 1} = (x - y \sqrt{k^2 + 1})(2k^2 + 1 + 2k \sqrt{k^2 + 1})
\]
\[
X_3 + K_3 \sqrt{y^2 + 1} = (x + k \sqrt{y^2 + 1})(2y^2 + 1 + 2y \sqrt{y^2 + 1}).
\]

(i) Taking norms gives \(X_i^2 - (Y_i^2 + 1)K_i^2 = Y_i^2.\)

(ii) \(gcd(X_i, Y_i) = gcd(x, y)\) and \(K_i > k\) for all \(i\).
Proposition. Type 1(a) solutions \((k_n, x_n, y)\),

\[ x_n + k_n \sqrt{D} = y(R + S\sqrt{D})^n, \quad y \geq 2, \]

where \(R = 2y^2 + 1\), \(S = 2y\), \(D = y^2 + 1\), can be expressed in terms of \(g_+\) and \(g_0\):

(i) \((k_1, x_1, y) = g_+(y, y, 0)\),

(ii) \((k_{n+1}, x_{n+1}, y) = g_0(k_n, x_n, y), \quad n \geq 1.\)
Generating the Type 1(b) solutions with $g_0$

Proposition. Type 1(b) solutions $(k_n, x_n, y)$,

$$x_n + k_n \sqrt{D} = (y^2 + \epsilon y + 1 + (y + \epsilon)\sqrt{y^2 + 1})(R + S\sqrt{D})^n,$$

where $R = 2y^2 + 1, S = 2y, D = y^2 + 1$, and where $y \geq 1$ if $\epsilon = 1$ and $y \geq 2$ if $\epsilon = -1$, can be expressed in terms of $g_+$ and $g_0$:

(i) $(k_1, x_1, y) = g_+(y, y^2 + \epsilon y + 1, y + \epsilon)$,

(ii) $(k_{n+1}, x_{n+1}, y) = g_0(k_n, x_n, y), n \geq 1.$
Proposition.

(i) Suppose that \((k, x, y)\) is an exceptional solution. Then \(g_+(k, x, y)\) and \(g_-(k, x, y)\) are Type 2 exceptional solutions.

(ii) Suppose that \((k, x, y)\) is a Type 2 exceptional solution. Then \(g_0(k, x, y)\) is also Type 2 exceptional solution.
This is constructed recursively from the trivial solutions

(i) \((t, t, 0), t \geq 2,\)
(ii) \((t, t^2 - t + 1, t - 1), t \geq 2,\)
(iii) \((t, t^2 + t + 1, t + 1), t \geq 1.\)

First apply \(g_+\) to each trivial solution, thereby producing a Type 1 exceptional solution. Then apply

\[ g_+ \quad (\uparrow), \quad g_0 \quad (\rightarrow), \quad g_- \quad (\downarrow) \]

recursively to each exceptional solution. In each case, this produces a tree of exceptional solutions \((k, x, y)\) in which \(\gcd(x, y)\) is constant. The Type 1 solutions are coloured red.
Example: Root node type \((t, t, 0), t \geq 2\)

Figure: Tree fragment starting from \((t, t, 0) = (2, 2, 0)\).
Example: Root node type \((t, t^2 - t + 1, t - 1), t \geq 2\)

Figure: Tree fragment starting from \((t, t^2 - t + 1, t - 1) = (2, 3, 1)\).
Example: Root node type \((t, t^2 + t + 1, t + 1), t \geq 1\)

Figure: Tree fragment with root node \((t, t^2 + t + 1, t + 1) = (1, 3, 2)\).
Example: Tree fragment of \((k(t), x(t), y(t))\) starting from 
\((t, t, 0)\)

\[
(16t^5 + 4t^3 + t, 32t^7 + 8t^5 + 6t^3 + t, 2t^2) \\
(2t^2, 2t^3 + t, t) \rightarrow (8t^4 + 4t^2, 8t^5 + 8t^3 + t, t) \\
(t, t, 0) \rightarrow (4t^3 - t, 8t^5 - 2t^3 + t, 2t^2)
\]
Example: \((k(t), x(t), y(t))\) from \((t, t^2 + t + 1, t + 1)\)

\[(k_1(t), x_1(t), y_1(t))\]
\[(4t^3 + 4t^2 + 3t + 1, 4t^4 + 4t^3 + 5t^2 + 3t + 1, t) \rightarrow (k_2(t), x_2(t), y_2(t))\]
\[(t, t^2 + t + 1, t + 1) \rightarrow (k_3(t), x_3(t), y_3(t))\]

\[k_1(t) = 64t^7 + 128t^6 + 176t^5 + 160t^4 + 104t^3 + 48t^2 + 15t + 2\]
\[x_1(t) = 256t^{10} + 768t^9 + 1408t^8 + 1792t^7 + 1712t^6 + 1264t^5 + 732t^4 + 324t^3 + 109t^2 + 25t + 3\]
\[y_1(t) = 4t^3 + 4t^2 + 3t + 1\]
\[k_2(t) = 16t^5 + 16t^4 + 20t^3 + 12t^2 + 5t + 1\]
\[x_2(t) = 16t^6 + 16t^5 + 28t^4 + 20t^3 + 13t^2 + 5t + 1\]
\[y_2(t) = t\]
\[k_3(t) = 16t^5 + 32t^4 + 36t^3 + 24t^2 + 9t + 2\]
\[x_3(t) = 64t^8 + 192t^7 + 320t^6 + 352t^5 + 272t^4 + 152t^3 + 61t^2 + 17t + 3\]
\[y_3(t) = 4t^3 + 4t^2 + 3t + 1.\]
Example: \((k(t), x(t), y(t))\) from \((t, t^2 - t + 1, t - 1)\)

\[
(4t^3-4t^2+3t-1,4t^4-4t^3+5t^2-3t+1,t) \rightarrow (k_2(t), x_2(t), y_2(t))
\]

\[
(t,t^2-t+1,t-1) \rightarrow (k_3(t), x_3(t), y_3(t))
\]

\[
k_1(t) = 64t^7-128t^6+176t^5-160t^4+104t^3-48t^2+15t-2
\]

\[
x_1(t) = 256t^{10}-768t^9+1408t^8-1792t^7+1712t^6-1264t^5+732t^4-324t^3+109t^2-25t+3
\]

\[
y_1(t) = 4t^3-4t^2+3t-1
\]

\[
k_2(t) = 16t^5-16t^4+20t^3-12t^2+5t-1
\]

\[
x_2(t) = 16t^6-16t^5+28t^4-20t^3+13t^2-5t+1
\]

\[
y_2(t) = t
\]

\[
k_3(t) = 16t^5-32t^4+36t^3-24t^2+9t-2
\]

\[
x_3(t) = 64t^8-192t^7+320t^6-352t^5+272t^4-152t^3+61t^2-17t+3
\]

\[
y_3(t) = 4t^3-4t^2+3t-1.
\]
All exceptional solutions are in the forest

This follows from:
Lemma. Let $E$ be the set of exceptional solutions $(K, X, Y)$. Then with $T = RK - SX$, where $R = 2Y^2 + 1$ and $S = 2Y$,

(i) $g_0$ maps $E$ 1–1 onto $\{(K, X, Y) \in E | Y + 1 < T\}$.

(ii) $g_+$ maps $E$ 1–1 onto $\{(K, X, Y) \in E | 0 < T < Y - 1\}$.

(iii) $g_-$ maps $E$ 1–1 onto $\{(K, X, Y) \in E | -(Y - 1) < T < 0\}$.

(iv) $g_+$ maps $\{(t, t, 0) | t \geq 2\}$ 1–1 onto $\{(K, X, Y) \in E | T = 0\}$.

(v) $g_+$ maps $\{(t, t^2 - t + 1, t - 1) | t \geq 2\}$ 1–1 onto $\{(K, X, Y) \in E | T = Y - 1\}$.

(vi) $g_+$ maps $\{(t, t^2 + t + 1, t + 1) | t \geq 1\}$ 1–1 onto $\{(K, X, Y) \in E | T = Y + 1\}$.
All exceptional solutions are in the forest

The following function $h$ takes an exceptional solution $(K, X, Y)$ and either produces another exceptional solution $(k, x, y)$ with $k < K$, or else creates a trivial solution.

$$h(K, X, Y) = \begin{cases} 
   g_0^{-1}(K, X, Y) & \text{if } Y + 1 < T \\
   g_+^{-1}(K, X, Y) & \text{if } 0 \leq T \leq Y + 1, T \neq Y \\
   g_-^{-1}(K, X, Y) & \text{if } -(Y - 1) < T < 0.
\end{cases}$$

Repeated application of $h$ will eventually lead to a trivial solution.
The exceptional solutions are polynomials in $t$

It is clear that the exceptional solutions have the form $(K(t), X(t), Y(t))$, where the components are polynomials in $t$ with integer coefficients, arising from the three types of root nodes:

(i) $(t, t, 0), t \geq 2$,
(ii) $(t, t^2 - t + 1, t - 1), t \geq 2$,
(iii) $(t, t^2 + t + 1, t + 1), t \geq 1$. 
Expressing $x$, $y$, $k$ in terms of $d$, $a$, $b$, $p$, $q$

Theorem. Suppose $(x, y)$ is a positive solution of Dujella’s equation $x^2 - (k^2 + 1)y^2 = k^2$. Let $d = \gcd(x + k, x - k)$ and define positive integers $a$ and $b$ by

$$a = \gcd((x + k)/d, k^2 + 1), \quad b = \gcd((x - k)/d, k^2 + 1).$$

Then

$$(x + k)/da = p^2, \quad (x - k)/db = q^2,$$

where $p$ and $q$ are integers. Also

(i) $x = d(ap^2 + bq^2)/2, \quad y = dpq,$

(ii) $ap^2 - bq^2 = 2k/d, \quad \gcd(p, q) = 1,$

(iii) $ab = k^2 + 1, \quad \gcd(a, b) = 1,$

(v) $k$ odd $\implies d$ even.
Restrictions on $a$, $b$ and $d$ for an exceptional solution

Proposition. If $(k, x, y)$ is an exceptional solution and $d = \gcd(x + k, x - k)$, then

(i) $d \neq k$, $d \neq 2k$,

(ii) $a > 2$, $b > 2$. 
Proposition. For an exceptional solution \((k, x, y)\), \(p\) and \(q\) satisfy the following inequalities:

\[
p^2 < \frac{(k^2 + 1)}{da}, \quad q^2 < \frac{(k - 1)^2}{db}.
\]

Hence \(p < k\) and \(q < k\).

Proof. If \((k, x, y)\) is an exceptional solution, then \(y < k - 1\), so \(x < k^2 - k + 1\). Hence

\[
p^2 = \frac{(x + k)}{da} < \frac{(k^2 + 1)}{da},
\]

\[
q^2 = \frac{(x - k)}{db} < \frac{(k - 1)^2}{db}.
\]
Connections with continued fractions

Proposition. Consider the equation

\[ ap^2 - bq^2 = \pm 2k/d, \]

where \( a < b \), \( D = ab = k^2 + 1 \), \( \gcd(a, b) = 1 = \gcd(p, q) \) and \( d \) divides \( 2k \). Let \( (P_m + \sqrt{D})/Q_m \) denote the \( m \)–th complete

quotient in the continued fraction expansion of \( \sqrt{D/a} = \sqrt{b/a} \),

with \( P_0 = 0 \) and \( Q_0 = a \).

(i) If \( d \geq 2 \), then \( p/q \) is a convergent \( A_m/B_m \) of \( \sqrt{b/a} \) and

\[ Q_{m+1} = 2k/d. \]

(ii) If \( d = 1 \), then \( p/q = (A_m + eA_{m-1})/(B_m + eB_{m-1}) \), where \( e = \pm 1 \). Also

\[ |Q_m - Q_{m+1} + 2eP_{m+1}| = 2k. \]
Some properties of the continued fraction of $\sqrt{b/a}$

Proposition. Suppose $1 < a < b$, $\gcd(a, b) = 1$, $ab = k^2 + 1$. Then the continued fraction of $\sqrt{b/a}$ is periodic:

$$\sqrt{b/a} = [a_0, a_1, \ldots, a_{l-1}, 2a_0].$$

and the period–length $l$ is odd. Also

(i) $A_{l-1}/B_{l-1} = k/a$.

(ii) $A_{l-2}/B_{l-2} = (b - ka_0)/(k - aa_0)$.

(iii) $A_{l}/B_{l} = (b + ka_0)/(k + aa_0)$. 
A parity conjecture

If \( ap^2 - bq^2 = 2k \) has a primitive solution \((p, q)\), where \( D = ab = k^2 + 1, \) \( k \) even, \( \gcd(a, b) = 1 \) and \( 2 < a < b \), then all \( Q_i \) are odd. Equivalently, using the identity

\[
Q_i Q_{i-1} = D - P_i^2,
\]

and the fact that if \( k \) is even, then \( D \) is odd, the conjecture is equivalent to the \( P_i \) being even. This in turn is equivalent to all partial quotients \( a_i \) being even, by virtue of the identity

\[
P_{i+1} = a_i Q_i - P_i.
\]
The unicity conjecture restated in terms of continued fractions

Conjecture. Consider the family of equations

\[ ap^2 - bq^2 = \pm 2k/d, \quad (1) \]

where \( d \) divides \( 2k \) (with \( d \) even if \( k \) is odd and \( d \neq k, d \neq 2k \)) and where \( \gcd(a, b) = 1, D = ab = k^2 + 1, 2 < a < b. \)

(i) Then there is at most one \((a, b, d)\) for which solubility occurs with \( \gcd(p, q) = 1. \)

(ii) In the case of solubility, there is exactly one solution \((p, q)\) with \( dpq < k - 1. \)
Example: $k = 8$

Here $D = k^2 + 1 = 65$ and only $(a, b, d) = (5, 13, 2)$ give solubility of $ap^2 - bq^2 = \pm 2k/d$ with $2 < a < b$, $ab = 65$, $\gcd(a, b) = 1$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$a_m$</th>
<th>$(P_m + \sqrt{D})/Q_m$</th>
<th>$A_m/B_m$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<td>$(0 + \sqrt{65})/5$</td>
<td>1/1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$(5 + \sqrt{65})/8$</td>
<td>2/1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$(3 + \sqrt{65})/7$</td>
<td>3/2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$(4 + \sqrt{65})/7$</td>
<td>5/3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$(3 + \sqrt{65})/8$</td>
<td>8/5</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
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</tr>
<tr>
<td>6</td>
<td>1</td>
<td>$(5 + \sqrt{65})/8$</td>
<td>29/18</td>
</tr>
</tbody>
</table>

$$5A_0^2 - 13B_0^2 = (-1)^1 Q_1 = -8 = -2k/d$$
$$5A_3^2 - 13B_3^2 = (-1)^4 Q_4 = 8 = 2k/d.$$
Example $k = 8$ continued

Then $(p_0, q_0) = (A_0, B_0) = (1, 1)$ is the smallest primitive solution of $5p^2 - 13q^2 = -8$, while $(p_1, q_1) = (A_3, B_3) = (5, 3)$ is the smallest primitive solution of $5p^2 - 13q^2 = 8$.

Also $(p_0, q_0)$ gives the unique exceptional solution of $x^2 - 65y^2 = 64$:

$$(x_0, y_0) = (d(ap_0^2 + bq_0^2)/2, dp_0q_0) = (18, 2).$$
$k = 12$

Here $D = k^2 + 1 = 145$ and only $(a, b, d) = (5, 29, 1)$ give solubility of $ap^2 - bq^2 = \pm 2k/d$ with $2 < a < b$, $ab = 145$, $\gcd(a, b) = 1$.

<table>
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<tr>
<th>$m$</th>
<th>$a_m$</th>
<th>$(P_m + \sqrt{D})/Q_m$</th>
<th>$A_m/B_m$</th>
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<td>4</td>
<td>2</td>
<td>$(10 + \sqrt{145})/9$</td>
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From the first period,

$$5(A_0 - A_{-1})^2 - 29(B_0 - B_{-1})^2 = (-1)^0(Q_0 - Q_1 - 2P_1) = -24 = -2k$$

$$5(A_2 + A_1)^2 - 29(B_2 + B_1)^2 = (-1)^2(Q_2 - Q_3 + 2P_3) = 24 = 2k.$$
Then \((p_0, q_0) = (A_0 - A_{-1}, B_0 - B_{-1}) = (1, 1)\) is the smallest primitive solution of \(5p^2 - 29q^2 = -24\), while
\((p_1, q_1) = (A_2 + A_1, B_2 + B_1) = (17, 7)\) is the smallest primitive solution of \(5p^2 - 29q^2 = 24\).

Also \((p_0, q_0)\) gives the unique exceptional solution of \(x^2 - 145y^2 = 144:\)

\[\left(x_0, y_0\right) = \left(d(ap_0^2 + bq_0^2)/2, dp_0q_0\right) = (17, 1).\]
An example from the forest

\[(k, x, y) = g_+(t, t^2 + t + 1, t + 1), \ t \geq 1. \text{ Then} \]

\[k = 4t^3 + 4t^2 + 3t + 1, \quad x = 4t^4 + 4t^3 + 5t^2 + 3t + 1, \quad y = t.\]

\[d = \begin{cases} 1 & \text{if } t \text{ is odd} \\ 2 & \text{if } t \text{ is even}, \end{cases} \]

\[a = \begin{cases} (4t^4 + 8t^3 + 9t^2 + 6t + 2)/2 & \text{if } t \text{ is even} \\ 4t^4 + 8t^3 + 9t^2 + 6t + 2 & \text{if } t \text{ is odd}, \end{cases} \]

\[b = \begin{cases} 8t^2 + 2 & \text{if } t \text{ is even} \\ 4t^2 + 1 & \text{if } t \text{ is odd}. \end{cases} \]
Forest example continued

(i) If $t$ is even,

$$\sqrt{b/a} = [0, t/2, 1, 1, t - 1, 1, 1, t - 1, 1, 1, t],$$  
period length 9.

$$p/q = A_1/B_1, \text{ where } A_1 = 1, B_1 = t/2.$$

(ii) If $t$ is odd,

$$\sqrt{b/a} = [0, t + 1, 2t, 2t, 2t + 2],$$  
period length 3.

$$p/q = (A_1 - A_0)/(B_1 - B_0), \text{ where } A_1 - A_0 = 1, B_1 - B_0 = t.$$
An example from deeper in the forest

\[(k, x, y) = g_{-g_{-g_{-g_+}}}(t, t, 0), \ t \geq 2. \ \text{Then}
\]

\[(k, x, y) = (16t^5 - 12t^3 + t, 128t^9 - 160t^7 + 56t^5 - 4t^3 + t, 8t^4 - 4t^2)\]

\[(d, a, b) = (2t, 16t^4 - 4t^2 + 1, 16t^6 - 20t^4 + 5t^2 + 1)\]

\[(p, q) = (2t^2 - 1, 2t).\]

Also

\[\sqrt{b/a} = [t - 1, 1, 2t - 2, 1, 2t - 1, 2t - 1, 1, 2t - 2, 1, 2t - 2].\]

period length 9 and \(Q_4 = 2k/d = 16t^4 - 12t^2 + 1, p/q = A_3/B_3.\)

\[
\begin{array}{c|cccccc}
  t & 2 & 3 & 4 & 5 & 6 \\
  k & 418 & 3567 & 15620 & 48505 & 121830 \\
\end{array}
\]
Some exact arithmetic BCmath programs

See
(i) http://www.numbertheory.org/php/dujella_test.html for a BCmath program which tests the unicity conjecture for a range of $k$ using the continued fraction of $\sqrt{b/a}$.

(ii) http://www.numbertheory.org/php/exceptionalforest.html for a BCmath program which enables one to guess the continued fraction corresponding to an exceptional node $(k(t), x(t), y(t))$. 