On a diophantine equation of Andrej Dujella

Keith Matthews

The unicity conjecture (Dujella 2009)

Let $k \geq 2, k \in \mathbb{N}$. Then the diophantine equation

$$x^2 - (k^2 + 1)y^2 = k^2$$

has at most one positive solution $(x, y)$ with $y < k - 1$. We call such a solution an *exceptional* solution.

Example. $k = 8$ is the first $k$ possessing an exceptional solution, namely $(x, y) = (18, 2)$.

We have verified the conjecture for $k \leq 2^{50}$. 
The conjecture has been proved in the following cases:

Filipin, Fujita and Mignotte:
(a) \( k^2 + 1 = p^n \) or \( 2p^n \), \( p \) an odd prime: no exceptional solutions.
(b) \( k = p^{2i} \) or \( p^{2i+1} \) or \( 2p^{2i+1} \), \( p \) an odd prime: no exceptional solutions.
(c) \( k = 2p^{2i} \), \( p \) an odd prime: the exceptional solution is \( (2p^{3i} + p^i, p^i) \).

Matthews and Robertson: \( k^2 + 1 = p^m q^n \) or \( 2p^m q^n \), \( m, n \geq 1 \), \( p \) and \( q \) distinct odd primes.
The $D(-1)$ 4–tuples conjecture

This states that there do not exist four positive integers such that the product of any two is one plus a square.
The unicity conjecture implies the $D(-1)$ 4–tuples conjecture (Dujella)

Assume the unicity conjecture and let $a, b, c, d$ be a $D(-1)$-quadruple with $0 < a < b < c < d$. Then $a = 1$ by Dujella-Fuchs (J. London Math. Soc. 2005) and hence

$$b = r^2 + 1, c = s^2 + 1, d = t^2 + 1.$$ 

Now consider the equation $(y^2 + 1)(t^2 + 1) = x^2 + 1$, i.e.,

$$x^2 - (t^2 + 1)y^2 = t^2.$$ 

By the conjecture, this diophantine equation has at most one solution with $0 < y < t - 1$.

But by assumption, it has at least two solutions with $0 < y < t$, namely, $y = r$ and $y = s$, and hence we must have $s = t - 1$.

However this contradicts a gap property (Dujella-Fuchs, Lemma 9) which implies that $d > c^2$, because the inequality

$$d = t^2 + 1 > c^2 = ((t - 1)^2 + 1)^2$$

does not hold for any $t > 2$. 
Type 1 and Type 2 solutions

Dujella’s equation can be written as

\[ x^2 - y^2 = (y^2 + 1)k^2. \]

We divide the exceptional solutions into two classes:

The Type 1 solutions are those for which \( y^2 + 1 \) divides \( x + y \) or \( x - y \), while Type 2 solutions are the remaining ones.

In the range \( k \leq 2^{50} \), there are 23,862,782 Type 1 and 73,034 Type 2 exceptional solutions.
Proposition. There is a 1–1 correspondence between the Type 1 solutions \((x, y)\), with \(x \equiv \epsilon y \pmod{y^2 + 1}\), \(\epsilon = \pm 1\) and the integer pairs \((r, s)\) which satisfy \(1 < r < s\) and

\[
\begin{align*}
  r^2 + s^2 &= k^2 + 1 \\
  s &\equiv \epsilon \pmod{r},
\end{align*}
\]

namely

\[
\begin{align*}
  r &= \frac{x - \epsilon y}{y^2 + 1}, \\
  s &= \frac{xy + \epsilon}{y^2 + 1},
\end{align*}
\]

where we take \(\epsilon = 1\) if \(y = 1\).

Example. \(k = 8, (x, y) = (18, 2), \epsilon = -1, (r, s) = (4, 7)\).
Example 1: Type 1(a) exceptional solution

These are the \((k_n, x_n, y)\), where

\[ x_n + k_n \sqrt{D} = y(R + S \sqrt{D})^n, \quad n \geq 1, \]

and \( R = 2y^2 + 1, \; S = 2y, \; D = y^2 + 1 \) and \( y \geq 2 \).

Here

\[ x_n \equiv (-1)^n y \pmod{y^2 + 1} \]

and \( y \) divides \( x_n \).
Example 2: Type 1(b) exceptional solution

These are the \((k_n, x_n, y)\), where

\[ x_n + k_n\sqrt{D} = (y^2 + \epsilon y + 1 + (y + \epsilon)\sqrt{D})(R + S\sqrt{D})^n, \ n \geq 1, \]

and where \(y \geq 1\) if \(\epsilon = 1\) and \(y \geq 2\) if \(\epsilon = -1\).

Here

\[ x_n \equiv (-1)^n \epsilon y \pmod{y^2 + 1}, \]

and \(\gcd(x_n, y) = 1\).
Theorem. If \((k, x, y)\) is a Type 1 solution, then

(i) either (a) \(y\) divides \(x\) and \(y > 1\), or (b) \(\gcd(x, y) = 1\).

(ii) \((k, x, y)\) is a Type 1(a) solution in case (a) and a Type 1(b) solution in case (b).
Producing exceptional solutions

The following three functions each create an exceptional solution $(K_i, X_i, Y_i)$ from an exceptional solution $(k, x, y)$:

(i) $g_+(k, x, y) = (K_1, X_1, Y_1)$, $Y_1 = k$,
(ii) $g_-(k, x, y) = g_+(k, x, -y) = (K_2, X_2, Y_2)$, $Y_2 = k$,
(iii) $g_0(k, x, y) = g_+(y, x, k) = (K_3, X_3, Y_3)$, $Y_3 = y$,

where

\[
X_1 + K_1 \sqrt{k^2 + 1} = (x + y \sqrt{k^2 + 1})(2k^2 + 1 + 2k \sqrt{k^2 + 1})
\]
\[
X_2 + K_2 \sqrt{k^2 + 1} = (x - y \sqrt{k^2 + 1})(2k^2 + 1 + 2k \sqrt{k^2 + 1})
\]
\[
X_3 + K_3 \sqrt{y^2 + 1} = (x + k \sqrt{y^2 + 1})(2y^2 + 1 + 2y \sqrt{y^2 + 1}).
\]

(i) Taking norms gives $X_i^2 - (Y_i^2 + 1)K_i^2 = Y_i^2$.
(ii) $\gcd(X_i, Y_i) = \gcd(x, y)$ and $K_i > k$ for all $i$. 
Generating the Type 1(a) solutions with $g_0$

Proposition. Type 1(a) solutions $(k_n, x_n, y)$,

$$x_n + k_n \sqrt{D} = y(R + S \sqrt{D})^n, y \geq 2,$$

where $R = 2y^2 + 1, S = 2y, D = y^2 + 1$, can be expressed in terms of $g_+$ and $g_0$:

(i) $(k_1, x_1, y) = g_+(y, y, 0)$,

(ii) $(k_{n+1}, x_{n+1}, y) = g_0(k_n, x_n, y), n \geq 1.$
Proposition. Type 1(b) solutions \((k_n, x_n, y)\),

\[ x_n + k_n\sqrt{D} = (y^2 + \epsilon y + 1 + (y + \epsilon)\sqrt{y^2 + 1})(R + S\sqrt{D})^n, \]

where \(R = 2y^2 + 1\), \(S = 2y\), \(D = y^2 + 1\), and where \(y \geq 1\) if \(\epsilon = 1\) and \(y \geq 2\) if \(\epsilon = -1\), can be expressed in terms of \(g_+\) and \(g_0\):

(i) \((k_1, x_1, y) = g_+(y, y^2 + \epsilon y + 1, y + \epsilon)\),

(ii) \((k_{n+1}, x_{n+1}, y) = g_0(k_n, x_n, y), n \geq 1.\)
Generating Type 2 exceptional solutions

Proposition.

(i) Suppose that \((k, x, y)\) is an exceptional solution. Then \(g^+(k, x, y)\) and \(g^-(k, x, y)\) are Type 2 exceptional solutions.

(ii) Suppose that \((k, x, y)\) is a Type 2 exceptional solution. Then \(g_0(k, x, y)\) is also Type 2 exceptional solution.
This is constructed recursively from the trivial solutions

(i) \((t, t, 0), t \geq 2,\)
(ii) \((t, t^2 - t + 1, t - 1), t \geq 2,\)
(iii) \((t, t^2 + t + 1, t + 1), t \geq 1.\)

First apply \(g_+\) to each trivial solution, thereby producing a Type 1 exceptional solution. Then apply

\[
g_+ \quad (\uparrow), \quad g_0 \quad (\rightarrow), \quad g_- \quad (\downarrow)
\]

recursively to each exceptional solution. In each case, this produces a tree of exceptional solutions \((k, x, y)\) in which \(\gcd(x, y)\) is constant. The Type 1 solutions are coloured \text{red}.\)
Example: Root node type \((t, t, 0)\), \(t \geq 2\)

Figure: Tree fragment starting from \((t, t, 0) = (2, 2, 0)\).
Example: Root node type \((t, t^2 - t + 1, t - 1), t \geq 2\)

Figure: Tree fragment starting from \((t, t^2 - t + 1, t - 1) = (2, 3, 1)\).
Example: Root node type \((t, t^2 + t + 1, t + 1), t \geq 1\)

Figure: Tree fragment with root node \((t, t^2 + t + 1, t + 1) = (1, 3, 2)\).
Example: Tree fragment of \((k(t), x(t), y(t))\) starting from \((t, t, 0)\)

\[
\begin{align*}
(2t^2, 2t^3 + t, t) & \rightarrow (8t^4 + 4t^2, 8t^5 + 8t^3 + t, t) \\

(t, t, 0) & \rightarrow (4t^3 - t, 8t^5 - 2t^3 + t, 2t^2)
\end{align*}
\]
Example: \((k(t), x(t), y(t))\) from \((t, t^2 + t + 1, t + 1)\)

\[
(4t^3 + 4t^2 + 3t + 1, 4t^4 + 4t^3 + 5t^2 + 3t + 1, t) \rightarrow (k_2(t), x_2(t), y_2(t))
\]

\[
(t, t^2 + t + 1, t + 1) \rightarrow (k_3(t), x_3(t), y_3(t))
\]

\[
k_1(t) = 64t^7 + 128t^6 + 176t^5 + 160t^4 + 104t^3 + 48t^2 + 15t + 2
\]

\[
x_1(t) = 256t^{10} + 768t^9 + 1408t^8 + 1792t^7 + 1712t^6 + 1264t^5 + 732t^4 + 324t^3 + 109t^2 + 25t + 3
\]

\[
y_1(t) = 4t^3 + 4t^2 + 3t + 1
\]

\[
k_2(t) = 16t^5 + 16t^4 + 20t^3 + 12t^2 + 5t + 1
\]

\[
x_2(t) = 16t^6 + 16t^5 + 28t^4 + 20t^3 + 13t^2 + 5t + 1
\]

\[
y_2(t) = t
\]

\[
k_3(t) = 16t^5 + 32t^4 + 36t^3 + 24t^2 + 9t + 2
\]

\[
x_3(t) = 64t^8 + 192t^7 + 320t^6 + 352t^5 + 272t^4 + 152t^3 + 61t^2 + 17t + 3
\]

\[
y_3(t) = 4t^3 + 4t^2 + 3t + 1.
\]
Example: \((k(t), x(t), y(t))\) from \((t, t^2 - t + 1, t - 1)\)

\[
\begin{align*}
& (k_1(t), x_1(t), y_1(t)) \\
& (4t^3 - 4t^2 + 3t - 1, 4t^4 - 4t^3 + 5t^2 - 3t + 1, t) \rightarrow (k_2(t), x_2(t), y_2(t)) \\
& (t, t^2 - t + 1, t - 1) \quad (k_3(t), x_3(t), y_3(t))
\end{align*}
\]

\[
\begin{align*}
k_1(t) &= 64t^7 - 128t^6 + 176t^5 - 160t^4 + 104t^3 - 48t^2 + 15t - 2 \\
x_1(t) &= 256t^{10} - 768t^9 + 1408t^8 - 1792t^7 + 1712t^6 - 1264t^5 + 732t^4 - 324t^3 + 109t^2 - 25t + 3 \\
y_1(t) &= 4t^3 - 4t^2 + 3t - 1 \\
k_2(t) &= 16t^5 - 16t^4 + 20t^3 - 12t^2 + 5t - 1 \\
x_2(t) &= 16t^6 - 16t^5 + 28t^4 - 20t^3 + 13t^2 - 5t + 1 \\
y_2(t) &= t \\
k_3(t) &= 16t^5 - 32t^4 + 36t^3 - 24t^2 + 9t - 2 \\
x_3(t) &= 64t^8 - 192t^7 + 320t^6 - 352t^5 + 272t^4 - 152t^3 + 61t^2 - 17t + 3 \\
y_3(t) &= 4t^3 - 4t^2 + 3t - 1.
\]
All exceptional solutions are in the forest

This follows from:
Lemma. Let $\mathcal{E}$ be the set of exceptional solutions $(K, X, Y)$. Then with $T = RK - SX$, where $R = 2Y^2 + 1$ and $S = 2Y$,

(i) $g_0$ maps $\mathcal{E}$ 1–1 onto $\{(K, X, Y) \in \mathcal{E} | Y + 1 < T\}$.

(ii) $g_+$ maps $\mathcal{E}$ 1–1 onto $\{(K, X, Y) \in \mathcal{E} | 0 < T < Y - 1\}$.

(iii) $g_-$ maps $\mathcal{E}$ 1–1 onto $\{(K, X, Y) \in \mathcal{E} | -(Y - 1) < T < 0\}$.

(iv) $g_+$ maps $\{(t, t, 0) | t \geq 2\}$ 1–1 onto $\{(K, X, Y) \in \mathcal{E} | T = 0\}$.

(v) $g_+$ maps $\{(t, t^2 - t + 1, t - 1) | t \geq 2\}$ 1–1 onto $\{(K, X, Y) \in \mathcal{E} | T = Y - 1\}$.

(vi) $g_+$ maps $\{(t, t^2 + t + 1, t + 1) | t \geq 1\}$ 1–1 onto $\{(K, X, Y) \in \mathcal{E} | T = Y + 1\}$.
All exceptional solutions are in the forest

The following function $h$ takes an exceptional solution $(K, X, Y)$ and either produces another exceptional solution $(k, x, y)$ with $k < K$, or else creates a trivial solution.

$$
h(K, X, Y) = \begin{cases} 
g_0^{-1}(K, X, Y) & \text{if } Y + 1 < T \\
g_+^{-1}(K, X, Y) & \text{if } 0 \leq T \leq Y + 1, \ T \neq Y \\
g_-^{-1}(K, X, Y) & \text{if } -(Y - 1) < T < 0.
\end{cases}$$

Repeated application of $h$ will eventually lead to a trivial solution.
The exceptional solutions are polynomials in $t$

It is clear that the exceptional solutions have the form $(K(t), X(t), Y(t))$, where the components are polynomials in $t$ with integer coefficients, arising from the three types of root nodes:

(i) $(t, t, 0)$, $t \geq 2$,
(ii) $(t, t^2 - t + 1, t - 1)$, $t \geq 2$,
(iii) $(t, t^2 + t + 1, t + 1)$, $t \geq 1$. 
Theorem. Suppose \((x, y)\) is a positive solution of Dujella’s equation \(x^2 - (k^2 + 1)y^2 = k^2\). Let \(d = \gcd(x + k, x - k)\) and define positive integers \(a\) and \(b\) by

\[
\begin{align*}
    a &= \gcd((x + k)/d, k^2 + 1), \\
    b &= \gcd((x - k)/d, k^2 + 1).
\end{align*}
\]

Then

\[
\begin{align*}
    (x + k)/da &= p^2, \\
    (x - k)/db &= q^2,
\end{align*}
\]

where \(p\) and \(q\) are integers. Also

(i) \(x = d(ap^2 + bq^2)/2, \quad y = dpq\),

(ii) \(ap^2 - bq^2 = 2k/d, \quad \gcd(p, q) = 1\),

(iii) \(ab = k^2 + 1, \quad \gcd(a, b) = 1\),

(v) \(k\) odd \(\implies d\) even.
Restrictions on $a$, $b$ and $d$ for an exceptional solution

Proposition. If $(k, x, y)$ is an exceptional solution and $d = \gcd(x + k, x - k)$, then

(i) $d \neq k$, $d \neq 2k$,
(ii) $a > 2$, $b > 2$. 
Proposition. For an exceptional solution \((k, x, y)\), \(p\) and \(q\) satisfy the following inequalities:

\[
p^2 < \frac{(k^2 + 1)}{da}, \quad q^2 < \frac{(k - 1)^2}{db}.
\]

Hence \(p < k\) and \(q < k\).

Proof. If \((k, x, y)\) is an exceptional solution, then \(y < k - 1\), so \(x < k^2 - k + 1\). Hence

\[
p^2 = \frac{(x + k)}{da} < \frac{(k^2 + 1)}{da},
\]

\[
q^2 = \frac{(x - k)}{db} < \frac{(k - 1)^2}{db}.
\]
Connections with continued fractions

Proposition. Consider the equation

\[ ap^2 - bq^2 = \pm 2k/d, \]

where \( a < b, \) \( D = ab = k^2 + 1, \) \( \gcd(a, b) = 1 = \gcd(p, q) \) and \( d \) divides \( 2k. \) Let \((P_m + \sqrt{D})/Q_m\) denote the \( m\)–th complete quotient in the continued fraction expansion of \( \sqrt{D/a} = \sqrt{b/a}, \) with \( P_0 = 0 \) and \( Q_0 = a.\)

(i) If \( d \geq 2, \) then \( p/q \) is a convergent \( A_m/B_m \) of \( \sqrt{b/a} \) and

\[ Q_{m+1} = 2k/d. \]

(ii) If \( d = 1, \) then \( p/q = (A_m + eA_{m-1})/(B_m + eB_{m-1}), \) where \( e = \pm 1. \) Also

\[ |Q_m - Q_{m+1} + 2eP_{m+1}| = 2k. \]
Proposition. Suppose $1 < a < b$, $\gcd(a, b) = 1$, $ab = k^2 + 1$. Then the continued fraction of $\sqrt{b/a}$ is periodic:

$$\sqrt{b/a} = [a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0}]$$

and the period–length $l$ is odd. Also

(i) $A_{l-1}/B_{l-1} = k/a$.
(ii) $A_{l-2}/B_{l-2} = (b - ka_0)/(k - aa_0)$.
(iii) $A_l/B_l = (b + ka_0)/(k + aa_0)$.
A parity conjecture

If \( ap^2 - bq^2 = 2k \) has a primitive solution \((p, q)\), where \( D = ab = k^2 + 1 \), \( k \) even, \( \gcd(a, b) = 1 \) and \( 2 < a < b \), then all \( Q_i \) are odd. Equivalently, using the identity

\[
Q_i Q_{i-1} = D - P_i^2,
\]

and the fact that if \( k \) is even, then \( D \) is odd, the conjecture is equivalent to the \( P_i \) being even. This in turn is equivalent to all partial quotients \( a_i \) being even, by virtue of the identity

\[
P_{i+1} = a_i Q_i - P_i.
\]
Conjecture. Consider the family of equations

\[ ap^2 - bq^2 = \pm 2k/d, \quad (1) \]

where \( d \) divides \( 2k \) (with \( d \) even if \( k \) is odd and \( d \neq k, d \neq 2k \)) and where \( \gcd(a, b) = 1, D = ab = k^2 + 1, 2 < a < b. \)

(i) Then there is at most one \((a, b, d)\) for which solubility occurs with \( \gcd(p, q) = 1. \)

(ii) In the case of solubility, there is exactly one solution \((p, q)\) with \( dpq < k - 1. \)
Example: $k = 8$

Here $D = k^2 + 1 = 65$ and only $(a, b, d) = (5, 13, 2)$ give solubility of $ap^2 - bq^2 = \pm 2k/d$ with $2 < a < b, ab = 65, \gcd(a, b) = 1.$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$a_m$</th>
<th>$(P_m + \sqrt{D})/Q_m$</th>
<th>$A_m/B_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$(0 + \sqrt{65})/5$</td>
<td>1/1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$(5 + \sqrt{65})/8$</td>
<td>2/1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$(3 + \sqrt{65})/7$</td>
<td>3/2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$(4 + \sqrt{65})/7$</td>
<td>5/3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$(3 + \sqrt{65})/8$</td>
<td>8/5</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>$(5 + \sqrt{65})/5$</td>
<td>21/13</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>$(5 + \sqrt{65})/8$</td>
<td>29/18</td>
</tr>
</tbody>
</table>

$5A_0^2 - 13B_0^2 = (-1)^1 Q_1 = -8 = -2k/d$

$5A_3^2 - 13B_3^2 = (-1)^4 Q_4 = 8 = 2k/d.$
Example $k = 8$ continued

Then $(p_0, q_0) = (A_0, B_0) = (1, 1)$ is the smallest primitive solution of $5p^2 - 13q^2 = -8$, while $(p_1, q_1) = (A_3, B_3) = (5, 3)$ is the smallest primitive solution of $5p^2 - 13q^2 = 8$.

Also $(p_0, q_0)$ gives the unique exceptional solution of $x^2 - 65y^2 = 64$:

$$(x_0, y_0) = (d(ap_0^2 + bq_0^2)/2, dp_0q_0) = (18, 2).$$
$k = 12$

Here $D = k^2 + 1 = 145$ and only $(a, b, d) = (5, 29, 1)$ give solubility of $ap^2 - bq^2 = \pm 2k/d$ with $2 < a < b$, $ab = 145$, \(\gcd(a, b) = 1\).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$a_m$</th>
<th>$(P_m + \sqrt{D})/Q_m$</th>
<th>$A_m/B_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>$(0 + \sqrt{145})/5$</td>
<td>2/1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$(10 + \sqrt{145})/9$</td>
<td>5/2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$(8 + \sqrt{145})/9$</td>
<td>12/5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$(10 + \sqrt{145})/5$</td>
<td>53/22</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>$(10 + \sqrt{145})/9$</td>
<td>118/49</td>
</tr>
</tbody>
</table>

From the first period,

\[
5(A_0 - A_{-1})^2 - 29(B_0 - B_{-1})^2 = (-1)^0(Q_0 - Q_1 - 2P_1) = -24 = -2k
\]

\[
5(A_2 + A_1)^2 - 29(B_2 + B_1)^2 = (-1)^2(Q_2 - Q_3 + 2P_3) = 24 = 2k.
\]
Example $k = 12$ continued

Then $(p_0, q_0) = (A_0 - A_{-1}, B_0 - B_{-1}) = (1, 1)$ is the smallest primitive solution of $5p^2 - 29q^2 = -24$, while $(p_1, q_1) = (A_2 + A_1, B_2 + B_1) = (17, 7)$ is the smallest primitive solution of $5p^2 - 29q^2 = 24$.

Also $(p_0, q_0)$ gives the unique exceptional solution of $x^2 - 145y^2 = 144$:

$$(x_0, y_0) = \left( \frac{d(ap_0^2 + bq_0^2)}{2}, dp_0q_0 \right) = (17, 1).$$
An example from the forest

\[(k, x, y) = g_+(t, t^2 + t + 1, t + 1), \; t \geq 1. \text{ Then}
\]

\[k = 4t^3 + 4t^2 + 3t + 1, \quad x = 4t^4 + 4t^3 + 5t^2 + 3t + 1, \quad y = t.\]

\[d = \begin{cases} 1 & \text{if } t \text{ is odd} \\ 2 & \text{if } t \text{ is even} \end{cases}, \]

\[a = \begin{cases} (4t^4 + 8t^3 + 9t^2 + 6t + 2)/2 & \text{if } t \text{ is even} \\ 4t^4 + 8t^3 + 9t^2 + 6t + 2 & \text{if } t \text{ is odd} \end{cases}, \]

\[b = \begin{cases} 8t^2 + 2 & \text{if } t \text{ is even} \\ 4t^2 + 1 & \text{if } t \text{ is odd}. \end{cases} \]
(i) If \( t \) is even,

\[
\sqrt{\frac{b}{a}} = [0, \frac{t}{2}, 1, 1, t - 1, 1, 1, t - 1, 1, 1, t], \text{ period length 9.}
\]

\[
p/q = A_1/B_1, \text{ where } A_1 = 1, B_1 = \frac{t}{2}.
\]

(ii) If \( t \) is odd,

\[
\sqrt{\frac{b}{a}} = [0, t + 1, 2t, 2t, 2t + 2], \text{ period length 3.}
\]

\[
p/q = (A_1 - A_0)/(B_1 - B_0), \text{ where } A_1 - A_0 = 1, B_1 - B_0 = t.
\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>12</td>
<td>55</td>
<td>154</td>
<td>333</td>
<td>616</td>
</tr>
</tbody>
</table>
An example from deeper in the forest

\[(k, x, y) = g \cdot g \cdot g \cdot g_+(t, t, 0), \quad t \geq 2.\] Then

\[(k, x, y) = (16t^5 - 12t^3 + t, 128t^9 - 160t^7 + 56t^5 - 4t^3 + t, 8t^4 - 4t^2)\]

\[(d, a, b) = (2t, 16t^4 - 4t^2 + 1, 16t^6 - 20t^4 + 5t^2 + 1)\]

\[(p, q) = (2t^2 - 1, 2t).\]

Also

\[\sqrt{b/a} = [t - 1, 1, 2t - 2, 1, 2t - 1, 2t - 1, 1, 2t - 2, 1, 2t - 2],\]

period length 9 and \(Q_4 = 2k/d = 16t^4 - 12t^2 + 1, p/q = A_3/B_3.\)

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>k</td>
<td>418</td>
<td>3567</td>
<td>15620</td>
<td>48505</td>
<td>121830</td>
</tr>
</tbody>
</table>
Some exact arithmetic BCmath programs

See
(i) http://www.numbertheory.org/php/dujella_test.html for a BCmath program which tests the unicity conjecture for a range of $k$ using the continued fraction of $\sqrt{b/a}$.

(ii) http://www.numbertheory.org/php/exceptionalforest.html for a BCmath program which enables one to guess the continued fraction corresponding to an exceptional node $(k(t), x(t), y(t))$.

(iii) http://www.numbertheory.org/php/dujella_minus.html for a BCmath program which tests the unicity conjecture by considering the equivalent diophantine equation $X^2 - (k^2 + 1)y^2 = -k^2$. 