The $d = mn$ Conjecture

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30 June, 2010

Abstract

Herein lies the proof of a conjecture of Keith’s, the truth of which would enable some code to be made faster!

Positive integers $m, n, d$ satisfy $m > n$ and $d = mn$. Define polynomials $A_d(q)$ and $B_{m,n}(q)$ by

$$A_d(q) = \sum_{k=0}^{d} a_k q^k \quad \text{and} \quad B_{m,n}(q) = \sum_{k=0}^{d} b_k q^k,$$

where

$$a_k = \binom{d}{k} \frac{(d+k-1)!}{(d-1)!},$$
$$b_k = \frac{m}{n} \binom{d-1}{k-1} \frac{(d+k-1)!}{(d-1)!}.$$

It is conjectured that $A_d(q) < B_{m,n}(q)$ for $q \geq 1$.

Let $N = n^2$. Some observations concerning the coefficients of the above polynomials are the following:

$$\begin{align*}
(k+1) a_{k+1} & = (d^2 - k^2) a_k, \\
kb_{k+1} & = (d^2 - k^2) b_k, \\
ika_k & = Nb_k, \\
N & = b_N. \tag{2}
\end{align*}$$

(1)

It follows from (2) that $b_k < a_k$ for $k < N$, while $b_k > a_k$ for $k > N$. We must now demonstrate that $B_{m,n}(q) > A_d(q)$ for $q \geq 1$, or equivalently, that for $q \geq 1$ we have

$$\begin{align*}
\sum_{k=N+1}^{d} (b_k - a_k) q^k & > \sum_{k=0}^{N-1} (a_k - b_k) q^k,
\end{align*}$$

where the sum has been split so that all terms are positive by the previous remark. Also, the $q^N$ term has been dropped since $a_N = b_N$. As $\sum_{k=N+1}^{d} (b_k - a_k) q^k \geq q^N \sum_{k=N+1}^{d} (b_k - a_k)$ and $q^N \sum_{k=0}^{N-1} (a_k - b_k) \geq \sum_{k=0}^{N-1} (a_k - b_k) q^k$ for $q \geq 1$, it will suffice to demonstrate that

$$\begin{align*}
\sum_{k=N+1}^{d} (b_k - a_k) > \sum_{k=0}^{N-1} (a_k - b_k), \tag{3}
\end{align*}$$

which corresponds to just showing that $B_{m,n}(1) > A_d(1)$.

Multiplying (3) by $N$ and applying (2), we see that this is equivalent to

$$\begin{align*}
\sum_{k=N+1}^{d} (k-N) a_k > \sum_{k=0}^{N-1} (N-k) a_k. \tag{4}
\end{align*}$$
By (1), for $0 \leq k < d$, we have

$$(k + 1)(a_{k+1} - a_k) = (d^2 - k^2)a_k - (k + 1)a_k = (d^2 - (k + 1)^2 + k)a_k > 0,$$

so the sequence $a_0, a_1, \ldots, a_d$ of coefficients is increasing. In particular,

$$\sum_{k=N+1}^{d} (k - N)a_k \geq \sum_{k=N+1}^{d} (k - N)a_{N+1} \quad \text{and} \quad \sum_{k=0}^{N-1} (N - k)a_{N-1} \geq \sum_{k=0}^{N-1} (N - k)a_k,$$

so it will suffice to show that

$$a_{N+1} \sum_{k=N+1}^{d} (k - N) > a_{N-1} \sum_{k=0}^{N-1} (N - k)$$

in order to prove (4) and thence the desired result.

The sums in (5) are $\sum_{k=N+1}^{d} (k - N) = (d - N)(d - N + 1)/2$ and $\sum_{k=0}^{N-1} (N - k) = N(N + 1)/2$. Also, from (1) we have $(N + 1)a_{N+1} = (d - N)(d + N)a_N$ and $Na_N = (d - N + 1)(d + N - 1)a_{N-1}$. Thus (5), after multiplying by the positive quantities $N+1$ and $(d - N + 1)(d + N - 1)$, dividing by $a_N/2$, and substituting the previous formulae for the sums, becomes

$$(d - N)^2(d - N + 1)^2(d + N)(d + N - 1) > N^2(N + 1)^2.$$  

To see that the inequality in (5) holds, notice that $d = mn$ and $N = n^2$, with $m \geq n + 1$, so $d \geq n^2 + n = N + n$. Thus,

$$(d - N)^2(d - N + 1)^2(d + N)(d + N - 1) \geq n^2(n + 1)^2(2N + n)(2N + n - 1) > n^2(n + 1)(N + 1)N = N^2(N + 1)^2.$$

It now follows that the prior inequalities also hold and, in particular, the conjecture is proved.