The $d = mn$ Conjecture

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Abstract

Herein lies the proof of a conjecture of Keith's, the truth of which would enable some code to be made faster!

Positive integers m, n, d satisfy $m > n$ and $d = mn$. Define polynomials $A_d(q)$ and $B_{m,n}(q)$ by

$$A_d(q) = \sum_{k=0}^{d} a_k q^k \quad \text{and} \quad B_{m,n}(q) = \sum_{k=0}^{d} b_k q^k,$$

where

$$\begin{aligned} a_k &= \binom{d}{k} \frac{(d+k-1)!}{(d-1)!} \\ b_k &= \frac{m}{n} \binom{d-1}{k-1} \frac{(d+k-1)!}{(d-1)!}. \end{aligned}$$

It is conjectured that $A_d(q) < B_{m,n}(q)$ for $q \geq 1$.

Let $N = n^2$. Some observations concerning the coefficients of the above polynomials are the following:

$$\begin{aligned} (k+1)a_{k+1} &= (d^2 - k^2)a_k && (1) \\ kb_{k+1} &= (d^2 - k^2)b_k \\ ka_k &= Nb_k && (2) \\ a_N &= b_N. \end{aligned}$$

It follows from (2) that $b_k < a_k$ for $k < N$, while $b_k > a_k$ for $k > N$. We must now demonstrate that $B_{m,n}(q) > A_d(q)$ for $q \geq 1$, or equivalently, that for $q \geq 1$ we have

$$\sum_{k=N+1}^{d} (b_k - a_k)q^k > \sum_{k=0}^{N-1} (a_k - b_k)q^k,$$

where the sum has been split so that all terms are positive by the previous remark. Also, the q^N term has been dropped since $a_N = b_N$. As $\sum_{k=N+1}^{d}(b_k - a_k)q^k \geq q^N \sum_{k=N+1}^{d}(b_k - a_k)$ and $q^N \sum_{k=0}^{N-1}(a_k - b_k) \geq \sum_{k=0}^{N-1}(a_k - b_k)q^k$ for $q \geq 1$, it will suffice to demonstrate that

$$\sum_{k=N+1}^{d} (b_k - a_k) > \sum_{k=0}^{N-1} (a_k - b_k), \tag{3}$$

which corresponds to just showing that $B_{m,n}(1) > A_d(1)$.

Mulitplying (3) by N and applying (2), we see that this is equivalent to

$$\sum_{k=N+1}^{d} (k - N)a_k > \sum_{k=0}^{N-1} (N - k)a_k. \tag{4}$$

By (1), for $0 \le k < d$, we have

$$(k+1)(a_{k+1} - a_k) = (d^2 - k^2)a_k - (k+1)a_k = (d^2 - (k+1)^2 + k)a_k > 0,$$

so the sequence $a_0, a_1, \dots, a_d$ of coefficients is increasing. In particular,

$$\sum_{k=N+1}^{d} (k-N)a_k \ge \sum_{k=N+1}^{d} (k-N)a_{N+1} \quad \text{and} \quad \sum_{k=0}^{N-1} (N-k)a_{N-1} \ge \sum_{k=0}^{N-1} (N-k)a_k,$$

so it will suffice to show that

$$a_{N+1} \sum_{k=N+1}^{d} (k-N) > a_{N-1} \sum_{k=0}^{N-1} (N-k) \tag{5}$$

in order to prove (4) and thence the desired result.

The sums in (5) are $\sum_{k=N+1}^{d}(k-N) = (d-N)(d-N+1)/2$ and $\sum_{k=0}^{N-1}(N-k) = N(N+1)/2$. Also, from (1) we have $(N+1)a_{N+1} = (d-N)(d+N)a_N$ and $Na_N = (d-N+1)(d+N-1)a_{N-1}$. Thus (5), after multiplying by the positive quantities $N+1$ and $(d-N+1)(d+N-1)$, dividing by $a_N/2$, and substituting the previous formulæ for the sums, becomes

$$(d-N)^2(d-N+1)^2(d+N)(d+N-1) > N^2(N+1)^2. \tag{6}$$

To see that the inequality in (5) holds, notice that $d = mn$ and $N = n^2$, with $m \ge n+1$, so $d \ge n^2 + n = N + n$. Thus,

$$\begin{aligned}
(d-N)^2(d-N+1)^2(d+N)(d+N-1) &\ge n^2(n+1)^2(2N+n)(2N+n-1) \\
&> n^2(n^2+1)(N+1)N \\
&= N^2(N+1)^2.
\end{aligned}$$

It now follows that the prior inequalities also hold and, in particular, the conjecture is proved.