

A CLASS OF GENERALIZED $3x + 1$ MAPPINGS OF BENOIT CLOITRE

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1. INTRODUCTION

In an email to the author dated July 25, 2011, Benoit Cloitre described a mapping $F_m : \mathbb{Z} \rightarrow \mathbb{Z}$ which is similar to the well-known $3x + 1$ mapping, where $m \geq 2$ is an integer. First $f_m(x)$ is defined by

$$(1) \quad f_m(x) = \left\lfloor \left(\frac{m+1}{m} \right) x \right\rfloor.$$

Then

$$(2) \quad F_m(x) = \begin{cases} \frac{f_m(x)}{2} & \text{if } f_m(x) \text{ is even} \\ \frac{3f_m(x)+1}{2} & \text{if } f_m(x) \text{ is odd.} \end{cases}$$

If $m = 3, 5$ or $m \geq 7$, all trajectories $x, F_m(x), F_m^{(2)}(x) = F_m(F_m(x)), \dots$, appear to eventually enter one of finitely many cycles; whereas if $m = 2, 4$ or 6 , there appear to be divergent trajectories (i.e., with $|F_m^{(n)}(x)| \rightarrow \infty$), but again only finitely many cycles.

F_m can be regarded as a $2m$ -branched generalized $3x + 1$ mapping ([1, p. 80]) for even m and an m -branched mapping for odd m . In both cases, we can use the Markov chain heuristics of [3] to predict everywhere cycling or the existence of divergent trajectories.

We remark that $0, 0$ and $-1, -1$ are always cycles, while $1, 2, 1$ is a cycle if $m \geq 3$ and $-4, -7, -4$ is a cycle if $m \geq 7$.

2. F_m AS A d -BRANCHED MAPPING

If $x = mK + i, 0 \leq i < m$, then (1) gives

$$(3) \quad \begin{aligned} f_m(x) &= \left\lfloor \left(\frac{m+1}{m} \right) (mK + i) \right\rfloor \\ &= (m+1)K + i + \left\lfloor \left(\frac{m+1}{m} \right) i \right\rfloor \\ &= (m+1)K + i. \end{aligned}$$

If m is odd, then (3) gives $f_m(x) \equiv i \pmod{2}$ and (2) implies

$$(4) \quad F_m(x) = \begin{cases} \frac{(m+1)K+i}{2} & \text{if } 0 \leq i < m, i \text{ even} \\ \frac{3(m+1)K+3i+1}{2} & \text{if } 0 \leq i < m, i \text{ odd} \end{cases} \\ = \begin{cases} (\frac{(m+1)}{2}x - \frac{i}{2})/m & \text{if } 0 \leq i < m, i \text{ even} \\ (\frac{3(m+1)}{2}x - \frac{3i-m}{2})/m & \text{if } 0 \leq i < m, i \text{ odd.} \end{cases}$$

Hence F_m is a d -branched mapping of the form

$$(5) \quad F_m(x) = (m_i x - r_i)/d \text{ if } x \equiv i \pmod{d}, 0 \leq i < d,$$

where $d = m$ and

$$(6) \quad m_i = \begin{cases} (m+1)/2 & \text{if } 0 \leq i < m, i \text{ even} \\ 3(m+1)/2 & \text{if } 0 \leq i < m, i \text{ odd.} \end{cases}$$

If m is even, we write $x = 2mK + i, 0 \leq i < 2m$ and (1) gives

$$(7) \quad f_m(x) = \begin{cases} 2(m+1)K + i & \text{if } 0 \leq i < m \\ 2(m+1)K + i + 1 & \text{if } m \leq i < 2m. \end{cases}$$

Hence

$$f_m(x) \equiv \begin{cases} i \pmod{2} & \text{if } 0 \leq i < m \\ i + 1 \pmod{2} & \text{if } m \leq i < 2m \end{cases}$$

and (2) implies

$$(8) \quad F_m(x) = \begin{cases} ((m+1)x - i)/2m & \text{if } 0 \leq i < m, i \text{ even} \\ (3(m+1)x + m - 3i)/2m & \text{if } 0 \leq i < m, i \text{ odd} \\ (3(m+1)x + 4m - 3i)/2m & \text{if } m \leq i < 2m, i \text{ even} \\ ((m+1)x + m - i)/2m & \text{if } m \leq i < 2m, i \text{ odd} \end{cases}$$

and F_m is a d -branched mapping of the form (5), where $d = 2m$ and

$$(9) \quad m_i = \begin{cases} m+1 & \text{if } 0 \leq i < m, i \text{ even} \\ 3(m+1) & \text{if } 0 \leq i < m, i \text{ odd} \\ 3(m+1) & \text{if } m \leq i < 2m, i \text{ even} \\ m+1 & \text{if } m \leq i < 2m, i \text{ odd.} \end{cases}$$

3. THE CASE OF m NOT DIVISIBLE BY 3

If 3 does not divide m , we see from (6) and (9) that $\gcd(m_i, d) = 1$ for $0 \leq i < d$ and F_m is of relatively prime type ([1, p. 82]). Consequently we expect

- (i) all trajectories will eventually enter a cycle if $m_0 \cdots m_{d-1} < d^d$;
- (ii) almost all trajectories will be divergent if $m_0 \cdots m_{d-1} > d^d$;
- (iii) the number of cycles will be finite.

Then if m is even, with $d = 2m$,

$$\begin{aligned} m_0 \cdots m_{d-1} < d^d &\iff (m+1)^m (3(m+1))^m < (2m)^{2m} \\ &\iff 3^m (m+1)^{2m} < (2m)^{2m} \\ &\iff 3(m+1)^2 < (2m)^2 \\ &\iff 0 < m^2 - 6m - 3 \\ &\iff 8 \leq m, \end{aligned}$$

while if m is odd, with $d = m$,

$$\begin{aligned} m_0 \cdots m_{d-1} < d^d &\iff \left(\frac{m+1}{2}\right)^{\frac{m+1}{2}} \left(\frac{3(m+1)}{2}\right)^{\frac{m-1}{2}} < m^m \\ &\iff (1 + 1/m)^m 3^{\frac{m-1}{2}} < 2^m. \end{aligned}$$

Now $(1 + 1/m)^m < 3$ so

$$\begin{aligned} (1 + 1/m)^m 3^{\frac{m-1}{2}} &< 3^{\frac{m+1}{2}} \\ &< 2^m, \text{ if } m \geq 5. \end{aligned}$$

Hence if m is not divisible by 3, we expect

- (i) every trajectory will eventually enter one of finitely many cycles if $m \geq 8$ is even or $m \geq 5$ is odd;
- (ii) almost all trajectories will be divergent if $m = 2$ or 4.

Example 1. $m = 2$. We find experimentally, using the author's CALC program [2], only the cycles $0, 0$ and $-1, -1$. Also the trajectories starting with 1 and -7 appear to be divergent.

Example 2. $m = 8$. We find experimentally the following eight cycles :

- (i) $0, 0$
- (ii) $-1, -1$
- (iii) $1, 2, 1$
- (iv) $215, 362, \dots, 383, 215$ (length 168)
- (v) $680, 1148, \dots, 1209, 680$ (length 21)
- (vi) $595, 1004, \dots, 1058, 595$ (length 21)
- (vii) $663, 1118, \dots, 1179, 663$ (length 21)
- (viii) $49868, 84152, \dots, 88655, 49868$ (length 21)

4. THE CASE OF m DIVISIBLE BY 3

Formulae (6) and (9) reveal that

$$\gcd(m_i, d) = \gcd(m_i, d^2) = \begin{cases} 1 & \text{if } m \text{ is odd and } m_i = (m+1)/2, \\ & \text{or } m \text{ is even and } m_i = m+1 \\ 3 & \text{if } m \text{ is odd and } m_i = 3(m+1)/2, \\ & \text{or } m \text{ is even and } m_i = 3(m+1). \end{cases}$$

Hence we are not dealing with a relatively prime mapping, but instead, one discussed in [3], where the behaviour of trajectories is determined by a Markov matrix

Experimentally, we found that when m is a multiple of 3, the stationary vector of $Q(d)$ is proportional to $X = (x_0, x_1, x_2, \dots, x_{d-3}, x_{d-2}, x_{d-1})^t$, where for $0 \leq t < d/3$,

$$(x_{3t}, x_{3t+1}, x_{3t+2}) = \begin{cases} (1, 1, 4) & \text{if } m = 6n \\ (2n+1, 2n+1, 8n+2) & \text{if } m = 12n+3 \\ (n+1, n+1, 4n+3) & \text{if } m = 12n+9. \end{cases}$$

Then for example, if $m = 12n+9$,

$$\begin{aligned} m_0^{x_0} \cdots m_{d-1}^{x_{d-1}} &= \overbrace{RSRS \cdots RS}^{2n+1} R \\ &= R^{2n+2} S^{2n+1}, \end{aligned}$$

where

$$\begin{aligned} R &= \left(\frac{m+1}{2}\right)^{n+1} \left(\frac{3(m+1)}{2}\right)^{n+1} \left(\frac{m+1}{2}\right)^{4n+3} \\ S &= \left(\frac{3(m+1)}{2}\right)^{n+1} \left(\frac{m+1}{2}\right)^{n+1} \left(\frac{3(m+1)}{2}\right)^{4n+3}. \end{aligned}$$

Then

$$\begin{aligned} &\left(\frac{m_0}{d}\right)^{x_0} \cdots \left(\frac{m_{d-1}}{d}\right)^{x_{d-1}} < 1 \\ &\iff m_0^{x_0} \cdots m_{d-1}^{x_{d-1}} < d^{x_0 + \cdots + x_{d-1}} \\ &\iff R^{2n+2} S^{2n+1} < m^{(4n+3)(6n+5)} \\ &\iff \left(\frac{m+1}{2}\right)^{(6n+5)(4n+3)} 3^{3n+2(4n+3)} < m^{(6n+5)(4n+3)} \\ &\iff ((1+1/m))^{(6n+5)} 3^{3n+2} < 2^{(6n+5)} \\ (15) \quad &\iff ((1+1/m))^{\frac{m+1}{2}} 3^{\frac{m-1}{2}} < 2^{\frac{m+1}{2}}. \end{aligned}$$

Now $(1+1/m)^m < 3$ and $1+1/m \leq 3/2$ if $m \geq 2$. Hence

$$\begin{aligned} (1+1/m)^{m+1} 3^{\frac{m-1}{2}} &< 3(1+1/m) 3^{\frac{m-1}{2}} \\ &< 3^{\frac{m+1}{2}} 3/2. \end{aligned}$$

Hence (15) holds if

$$3^{\frac{m+1}{2}} 3/2 < 2^{m+1}$$

or equivalently

$$3^{m+3} < 4^{m+2}$$

and this holds for $m \geq 2$.

Hence we expect all trajectories to eventually cycle. Also it seems certain that there are only finitely many cycles for each such m . We have found seven cycles when $m = 9$:

- (i) 0, 0
- (ii) -1, -1
- (iii) 1, 2, 1
- (iv) -4, -7, -4
- (v) -6, -10, -6

- (vi) $-11, -19, -11$
- (vii) $14, 23, 38, 21, 35, 19, 32, 53, 29, 16, 26, 14$.

REFERENCES

- [1] J.C. Lagarias, Ed. *The Ultimate Challenge: The $3x + 1$ Problem*, AMS 2011.
- [2] K. R. Matthews, http://www.numbertheory.org/calc/krm_calc.html, CALC, a number theory calculator.
- [3] K. R. Matthews and A. M. Watts, *A Markov approach to the generalized Syracuse algorithm*, *ibid.* 45 (1985), 29–42.