Some properties of the continued fraction expansion of \((m/n) e^{1/q}\)

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Some properties of the continued fraction expansion of \((m/n)e^{1/q}\)

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Introduction. Continued fractions of the form

\[
\left[ b_1, \ldots, b_h, f_1(x), \ldots, f_k(x) \right]_{x=0}^\infty
\]

are called Hurwitzian if \(b_1, \ldots, b_h, f_1(x), \ldots, f_k(x)\) are polynomials with rational coefficients which take positive integral values for \(x = 0, 1, 2, \ldots\), and at least one of the polynomials is not constant. \(f_1(x), \ldots, f_k(x)\) are said to form a quasi-period.

The expansions

\[
e = \left[ 2, 1, 2x+2, 1 \right]_{x=0}^\infty \quad \text{and} \quad e^{1/q} = \left[ 1, (2x+1)q-1, 1 \right]_{x=0}^\infty,
\]

where \(q\) is a positive integer, \(q > 1\), are well-known examples. (See Perron (4), Davis (1), or Walters (5).)

If \(m\) and \(n\) are coprime positive integers, it follows from a theorem of A. Hurwitz (see Perron (4)) that \((m/n)e^{1/q}\) also has a Hurwitzian continued fraction with a quasi-period consisting of linear progressions, apart from constants.

In this paper we prove the existence of a quasi-period containing exactly \(mn\) linear progressions as above, and a necessary and sufficient condition for each of these progressions to have the form \(2qx + b\) is also derived.

The proofs are based on matrix methods developed by Kolden (2) and Walters (5).

Notation. Let \(\{A_r\}\) be a sequence of non-singular matrices with real number elements, and let

\[
\prod_{r=0}^N A_r = \begin{pmatrix} P_N & R_N \\ Q_N & S_N \end{pmatrix}.
\]

Then we write \(x \sim \prod_{r=0}^\infty A_r\) if each of the sequences \(p_N/q_N\) and \(r_N/s_N\) converges to \(x\), as \(N \to \infty\). Continued fractions and matrices are connected by the relation

\[
[a_0, a_1, \ldots] \sim \prod_{r=0}^\infty U_{a_r}, \quad \text{where} \quad U_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.
\]

Any non-empty product \(U_{b_1}U_{b_2} \ldots U_{b_k}, b_i > 0\), is denoted by \(P\). Kolden proved that any matrix \(M\) with non-negative integer elements and determinant \(\pm 1\) (apart from \(U_0\) and the identity matrix) has one of the forms \(P, U_0P, PU_0, U_0PU_0\). The factorization is unique. (See (2) pages 159–161.)
The following facts are used frequently:

(i) \( PU_0 P = P, \ U_0 P U_0 = U_0 P, \ MP = P \) or \( U_0 P, \ MU_0 P = P \) or \( U_0 P \). (These follow from \( U_0 U_0 U_b = U_{a+b} \).)

(ii) If \( P = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \) then \( p \geq r \) and \( q \geq s \).

Lemma 6 of Walters' (5) is used twice. It implies, under conditions satisfied here, that

\[
A_0 \prod_{r=1}^\infty A_r \sim \prod_{r=1}^\infty (A_0 A_r A_0^{-1}).
\]

Our investigation is based on a result of Lehmer (3).

**Lemma 1.**

Proof. An easy induction shows the matrix product is

\[
A_d(q) B_d(q) \begin{pmatrix} \frac{2(r-1)q+1}{2(r-1)q} & \frac{2r-1}{2r-1} \\ 0 & (-1)^d \end{pmatrix} \mod d.
\]

The lemma follows, since \( (d + k - 1)!/(d - 1)! \) is divisible by \( d \) if \( k \geq 1 \), while \( (d + k)!/(d - 1)! \) is divisible by \( d \) if \( k \geq 0 \).

**Definition.** Polynomials in \( x \) are defined by

\[
\begin{pmatrix} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{pmatrix} = \prod_{r=1}^d \begin{pmatrix} (2dx+r-1)q+1 & (2dx+r-1)q \\ (2dx+r-1)q & (2dx+r-1)q-1 \end{pmatrix}.
\]

Then by Lemma 1 the coefficients of \( G_d(x) \) and \( R_d(x) \) are divisible by \( d \).

**Lemma 2.** Let \( d = mn \). Then

\[
e_{1/2} \sim \prod_{x=0}^\infty \begin{pmatrix} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{pmatrix},
\]

where the matrices are unimodular with integer elements, positive unless \( m = n = q = 1 \).

**Proof.** We have

\[
e_{1/2} \sim \prod_{y=0}^\infty \begin{pmatrix} (2y+1)q+1 & (2y+1)q \\ (2y+1)q & (2y+1)q-1 \end{pmatrix},
\]

by Theorem 1 and Lemma 2 of (5).
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Consequently, by Lemma 1 of (5),

$$\frac{m}{n} e^{1/q} \sim \left(\begin{array}{c} m \\ n \end{array}\right) \prod_{x=0}^{\infty} \left(\begin{array}{cc} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{array}\right).$$

Our Lemma 2 then follows from the equation

$$\left(\begin{array}{cc} F_d(x) \\ (n/m) G_d(x) \\ S_d(x) \end{array}\right) = \left(\begin{array}{cc} m & 0 \\ 0 & n \end{array}\right) \left(\begin{array}{cc} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{array}\right) \left(\begin{array}{cc} m & 0 \\ 0 & n \end{array}\right)^{-1}, \ldots \tag{1}$$

either by direct comparison of convergents or by using Lemma 6 of (5).

The factorization of the matrices occurring in Lemma 2 must now be examined. Our method is based essentially on the proof of Hurwitz’s theorem presented in Perron (4), pages 110–123.

**Lemma 3.** Let $A = \left(\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}\right)$ have non-negative integer elements, with $a_1 a_3 \neq 0$, $\Delta = a_1 a_4 - a_2 a_3 \neq 0$ and either $a_1 \geq a_2$ or $a_3 \geq a_4$. Then $A$ may be factorized uniquely as

$$A = BC,$$

where $B$ is of type $P$ or $U_0 P$ and

$$C = \left(\begin{array}{cc} p & r \\ 0 & s \end{array}\right),$$

with $p$ and $s$ positive integers, and $r$ an integer.

If $B = \left(\begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array}\right)$, then $\det B = \Delta/|\Delta|$, $p = (a_1, a_3)$, $b_1 = a_1/p$, $b_3 = a_3/p$, $ps = |\Delta|$ and $-s < r < p$. (See Perron (4) page 111 for the proof of a similar result.)

The matrix $A_x$ defined by

$$A_x = \left(\begin{array}{cc} m & 0 \\ 0 & n \end{array}\right) \left(\begin{array}{cc} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{array}\right) \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right)$$

is evidently of type $A$ for integers $x \geq 0$, and we can obtain the following factorization:

**Lemma 4.**

$$A_x = \left(\begin{array}{cc} F_d(x) \\ (n/m) G_d(x) \\ S_d(x) \end{array}\right) B_0 C_1,$$

where $B_0 C_1$ is the factorization of

$$\left(\begin{array}{cc} m & 0 \\ 0 & n \end{array}\right) \left(\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right).$$

**Proof.**

$$A_x = \left(\begin{array}{cc} m & 0 \\ 0 & n \end{array}\right) \left(\begin{array}{cc} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{array}\right) \left(\begin{array}{cc} m & 0 \\ 0 & n \end{array}\right)^{-1} \left(\begin{array}{cc} m & 0 \\ 0 & n \end{array}\right) \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc} F_d(x) \\ (n/m) G_d(x) \\ S_d(x) \end{array}\right) \left(\begin{array}{cc} m & 0 \\ 0 & n \end{array}\right) \left(\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right), \text{ by equation } (1).$$

**Lemma 5.** Matrices $B_t$ and $C_{t+1}$ are defined by the recurrence relation

$$B_t C_{t+1} = Q_t^{-1} C_t U_{a-1} U_1 U_1 \quad (t = 1, 2, \ldots, d),$$
where \( a = \{2(dx + t) - 1\}q \) and \( Q_t = Q_t(x) = U_k \), where \( k = p_t(2qx - 1) \). Here \( B_tC_{t+1} \) is the factorization of

\[
\begin{pmatrix}
2s_t \\
p_t(2d - 1 + (4t - 2)q) + 2r_t \\
p_t(d + (2t - 1)q) + r_t
\end{pmatrix}
\]

and \( C_1 \) is defined in Lemma 4. Then the factorization

\[
A_x = B_0 \left( \prod_{t=1}^d Q_t B_t \right) C_{d+1}
\]

holds.

**Proof.** With an obvious simplified notation, noting that

\[
\begin{pmatrix}
a + 1 \\
a \\
a - 1
\end{pmatrix} = U_1 U_{a-1} U_1,
\]

we have

\[
A_x = B_0 C_1 U_1^{-1} \begin{pmatrix}
F \\
G \\
S
\end{pmatrix} U_1 = B_0 C_1 U_1^{-1} \begin{pmatrix}
(a + 1) & a \\
a & a - 1
\end{pmatrix} U_1
\]

\[
= B_0 C_1 U_1^{-1} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} U_1 = B_0 C_1 \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

by the recurrence relation;

\[
B_0 C_1 \Pi(C_t^{-1} Q_t B_t C_{t+1})
\]

**Lemma 6.**

\[
\begin{pmatrix}
F_d(x) \\
(n/m) G_d(x) \\
S_d(x)
\end{pmatrix} = B_0 \left( \prod_{t=1}^d Q_t B_t \right) B_0^{-1}.
\]

**Proof.** Lemmas 4 and 5 give

\[
A_x = \begin{pmatrix}
F_d(x) \\
(n/m) G_d(x) \\
S_d(x)
\end{pmatrix} B_0 C_1 = B_0 \left( \prod_{t=1}^d Q_t B_t \right) C_{d+1}.
\]

Using remarks (i) and (ii) of the introduction, it is not hard to see that the uniqueness conditions of Lemma 3 are satisfied if \( x \geq 1 \). Hence \( C_1 = C_{d+1} \). Multiplication on the right by \( C_1^{-1} B_0^{-1} \) then gives Lemma 6.

**Theorem 1.**

\[
\left( \frac{m}{\pi} \right) e^{\frac{1}{\pi}} \sim B_0 \prod_{x=0}^\infty \prod_{t=1}^\infty Q_t(x) B_t
\]

\[
\frac{m}{\pi} e^{\frac{1}{\pi}} \sim \begin{pmatrix}
F_d(0) \\
(n/m) G_d(0) \\
S_d(0)
\end{pmatrix} B_0 \prod_{x=1}^\infty \prod_{t=1}^d Q_t(x) B_t.
\]

**Proof.** By Lemmas 2 and 6, and Lemma 6 of (5)

\[
\frac{m}{\pi} e^{\frac{1}{\pi}} \sim \prod_{x=0}^\infty \left( B_0 \left( \prod_{t=1}^d Q_t(x) B_t \right) B_0^{-1} \right) = B_0 \prod_{x=0}^\infty \prod_{t=1}^d Q_t(x) B_t.
\]

The proof of (b) is similar.

**Remarks.** (i) The matrices of (b) are of type \( M \) and may be factorized, as mentioned in the introduction. It follows that the regular continued fraction for \( (m/n) e^{\frac{1}{\pi}} \) has a quasi-period containing exactly \( mn \) linear progressions.

(ii) The expansion (a) does not give the regular continued fraction for \( (m/n) e^{\frac{1}{\pi}} \) as the matrix \( Q_t(0) \) has a negative element.
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As a simple example we consider $3e$. It is easily verified that

$$p_1 = p_2 = p_3 = 1, \quad B_0 = U_3, B_1 = B_2 = U_0 U_1 U_2 U_1, \quad B_3 = U_0 U_1 U_2 U_5,$$

$$\begin{pmatrix} F_3(0) & 3R_3(0) \\ 13G_3(0) & S_3(0) \end{pmatrix} = \begin{pmatrix} 106 & 261 \\ 13 & 32 \end{pmatrix} = U_3 U_5 U_3 U_5.$$

Theorem 1 (b) then gives

$$3e \sim U_8 U_9 U_2 U_2 U_0 U_3 \prod_{x=1}^{\infty} \{U_{2x-1} U_0 U_1 U_3 U_1 \} \{U_{2x-1} U_0 U_1 U_5 U_1 \} \{U_{2x-1} U_0 U_1 U_3 U_1 \}$$

$$\sim U_8 U_9 U_2 U_5 \prod_{x=1}^{\infty} \{U_{2x} U_3 U_5 U_1 U_2 U_2 U_0 U_3\},$$

and the corresponding regular continued fraction is

$$3e = [8, 6, 2, 5, 1, 2x, 5, 1, 2x, 5, 1].$$

It is natural to ask when the numbers $p_t$ of Theorem 1 satisfy $p_t = 1$ for $t = 1, 2, \ldots, d$. This is answered by Theorem 2. It is convenient to state, without proof, the following lemma:

**Lemma 7.** If $\begin{pmatrix} X & V \\ Y & W \end{pmatrix}$ is a matrix of type $P$ or $U_0 P$, with $XW - YV = 1$ and $X > 1$, then $V$ is determined by the conditions $YV \equiv -1 \pmod X$ and $0 < V < X$.

**Theorem 2.** Each of the progressions $p_t^2(2qx - 1)$ in Theorem 1 reduces to $2qx - 1$ if, and only if, the integers $K_t$ defined by the recurrence relation

$$K_t = (4t - 2)qK_{t-1} + K_{t-2} \quad \text{for} \quad 2 \leq t \leq d,$$

with $K_0 = 1, K_1 = 2q - 1 + 2r_1$, satisfy $(K_t, d) = 1$.

**Proof.** (i) Assume $p_t = 1$ for $t = 1, 2, \ldots, d$. Then the definition of $B_tC_{t+1}$ in Lemma 5 gives

$$\begin{pmatrix} 2d \\ 2d - 1 + (4t - 2)q + 2r_t \end{pmatrix} \begin{pmatrix} d \\ d + (2t - 1)q + r_t \end{pmatrix} = B_tC_{t+1}.$$

Hence

$$B_t = \begin{pmatrix} 2d \\ 2d - 1 + (4t - 2)q + 2r_t \end{pmatrix} v_t, \quad C_{t+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} r_{t+1} \\ d \end{pmatrix}.$$

By Lemma 7, $v_t$ is determined by

$$\{2r_t + (4t - 2)q - 1\} v_t \equiv -1 \pmod {2d} \quad (2)$$

and $0 < v_t < 2d$.

The matrix equation obtained by equating the two expressions for $B_tC_{t+1}$ gives

$$1 = 2r_{t+1} + v_t. \quad (3)$$

Congruence (2) and equation (3) imply

$$\{(4t - 2)q - v_{t-1}\} v_t \equiv -1 \pmod {2d} \quad (4)$$

for $t = 2, \ldots, d$. 


Residue classes \( k_t \), each prime to \( 2d \), can now be defined by the recurrence relation

\[
k_t \equiv k_{t-1} \{(4t-2)q-v_{t-1}\} \pmod{2d} \quad (2 \leq t \leq d),
\]
with \( k_1 \equiv 2q+2r_1-1 \pmod{2d} \).

Congruence (4) may then be written as

\[
k_t v_t \equiv -k_{t-1} \pmod{2d}.
\]

This congruence is valid for \( t = 1 \) on taking \( k_0 \equiv 1 \pmod{2d} \). Substituting for \( k_{t-1} v_{t-1} \) in congruence (5) gives

\[
k_t = (4t-2)qk_{t-1} + k_{t-2} \pmod{2d},
\]
completing the ‘only if’ part of the proof of Theorem 2.

(ii) Assume the integers \( K_t \), defined in Theorem 2, satisfy

\[
(K_t, d) = 1 \quad \text{for } t = 1, 2, \ldots, d.
\]

Since \( K_t \) is clearly odd, \( (K_t, 2d) = 1 \).

By definition, \( B_t C_{t+1} \) is the factorization of

\[
\left( \frac{p_t(2d-1+(4t-2)q)+2r_t}{2s_t} \right) \left( \frac{p_t(d+(2t-1)q)+r_t}{s_t} \right).
\]

Also

\[
B_t = \left( \begin{array}{cc} X_t & V_t \\ Y_t & W_t \end{array} \right),
\]

where

\[
X_t = 2s_t/p_{t+1}, \quad Y_t = \left[ p_t(2d-1+(4t-2)q)+2r_t \right]/p_{t+1},
\]
and

\[
p_{t+1} = (2s_t, p_t(2d-1+(4t-2)q)+2r_t).
\]

By Lemma 7, \( V_t \) is determined by

\[
Y_t V_t \equiv -1 \pmod{X_t} \quad \text{and} \quad 0 < V_t < X_t.
\]

Since \( (K_t, 2d) = 1 \) for \( t = 0, 1, \ldots, d \), we can define integers \( v_t \) by

\[
K_t v_t \equiv -K_{t-1} \pmod{2d} \quad \text{and} \quad 0 < v_t < 2d
\]

for \( t = 1, 2, \ldots, d \).

It is now shown that \( v_t = V_t \) and \( p_t = 1 \), for \( t = 1, 2, \ldots, d \). We use induction and assume

\[
p_1 = 1, p_2 = 1, \ldots p_t = 1 \quad \text{and} \quad V_1 = v_1, \ldots, V_{t-1} = v_{t-1}
\]

for some \( t \) with \( 2 \leq t < d \).

We first verify that the assumption is correct for \( t = 2 \).

By definition, \( p_1 = (m, n) = 1 \). Hence \( s_1 = d \) and

\[
B_1 C_2 = \left( \begin{array}{cc} 2d & d \\ 2d-1+2q+2r_1 & d+q+r_1 \end{array} \right).
\]

Consequently \( p_2 = (2d, 2d+K_1) = (2d, K_1) = 1 \),

and

\[
B_1 = \left( \begin{array}{cc} 2d & V_1 \\ 2d+K_1 & W_1 \end{array} \right),
\]
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where \((2d + K_1)V_1 \equiv -1 \pmod{2d}\) and \(0 < V_1 < 2d\). But \(K_1v_1 \equiv -1 \pmod{2d}\) and \(0 < v_1 < 2d\), so that \(v_1 = V_1\). This completes the verification for \(t = 2\).

To complete the induction, we need the following result which is easily deduced:

\[
\{(4t - 2)q - v_{t-1}\}v_t \equiv -1 \pmod{2d}
\]

for \(t \geq 2\).

It is now shown that \(p_{t+1} = 1\).

The induction hypothesis gives \(p_t = 1\) and \(s_t = d\), so that

\[
B_tC_{t+1} = \begin{pmatrix} 2d & d \\ 2d - 1 + (4t - 2)q + 2r_t & d + (2t - 1)q + r_t \end{pmatrix}
\]

and consequently \(p_{t+1} = (2d, (4t - 2)q + 2r_t - 1)\).

Also since \(p_{t-1} = 1\), \(s_{t-1} = d\), and \(v_{t-1} = V_{t-1}\), we have

\[
B_{t-1}C_t = \begin{pmatrix} 2d & d \\ 2d - 1 + (4t - 6)q + 2r_{t-1} & d + (2t - 3)q + r_{t-1} \end{pmatrix}
\]

Thus \(2r_t + v_{t-1} = 1\) and \(p_{t+1} = (2d, (4t - 2)q - v_{t-1})\).

Congruence (6) then implies \(p_{t+1} = 1\). Finally, we show that \(V_t = v_t\).

By definition

\[
B_t = \begin{pmatrix} 2d & V_t \\ 2d + (4t - 2)q - v_{t-1} & W_{t-1} \end{pmatrix},
\]

where by Lemma 7,

\[
\{2d + (4t - 2)q - v_{t-1}\}V_t \equiv -1 \pmod{2d} \quad \text{and} \quad 0 < V_t < 2d.
\]

Congruence (6) together with \(0 < v_t < 2d\), gives \(V_t = v_t\) and the induction is complete.

The following theorem deals with \(me^{1/q}\). Here \(r_1 = 0\), \(K_0 = 1\), and \(K_1 = 2q - 1\).

**Theorem 3.** Let \(q_t\) denote a prime with the property that \((K_t, q_t) = 1\) for \(t = 1, 2, \ldots, q_t\). Also let \(S\) be the set of natural numbers \(m\) for which each of the progressions \(p_t^2(2qx - 1)\) reduces to \(2qx - 1\), in the expansions of \(me^{1/q}\) given by Theorem 1. Then \(S\) is identical with \(T\), the set of numbers which contain only \(q_1, q_2, \ldots\) as prime factors.

**Proof.** (i) Suppose \(m\) belongs to \(S\). Then by Theorem 2, \((K_t, m) = 1\) for \(t = 1, 2, \ldots, m\). Hence \((K_t, q) = 1\) for \(t = 1, 2, \ldots, q\), for each prime \(q\) dividing \(m\).

(ii) Suppose \(m\) belongs to \(T\). Then to deduce that \(m\) belongs to \(S\), it suffices, by Theorem 2, to prove that the condition \((K_t, q) = 1\) for \(t = 1, 2, \ldots, q\) is equivalent to \((K_t, q) = 1\) for all \(t\).

However, this follows from the congruence

\[
K_{t+q} = (-1)^q\cdot K_t \pmod{q} \quad \text{for} \quad t \geq 0.
\]

This congruence is easily deduced from the formula

\[
K_t = \sum_{r=0}^{t} \binom{t}{r} (-1)^{t-r} \frac{(t+r)!}{t!} q^r,
\]

which may be verified by induction.
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