We are dealing with the polynomial \( f(x, y) = ax^2 + bxy + cy^2 \), where \( a > 0 \) and \( d = b^2 - 4ac > 0 \) is not a square. Let \( \rho = (-b + \sqrt{d})/2a \) and \( \sigma = (-b - \sqrt{d})/2a \) be the roots of \( f(x, 1) \). The following was mentioned by M. Pavone in 1986.

**Proposition 1.** (Lemma 5 of [3].) Let \( \rho = [a_0, \ldots, a_m, b_1, \ldots, b_n] \) and \( \sigma = [c_0, \ldots, c_r, d_1, \ldots, d_n] \) be the roots of \( f \). Also let \( p_h/q_h \) and \( P_h/Q_h \) denote the convergents of \( \rho \) and \( \sigma \), respectively. We do not require the periods to have minimal lengths, but assume \( m \) and \( r \) are minimal, i.e., \( a_m \neq b_n \) and \( c_r \neq d_n \). It is also convenient to assume \( n \geq 4 \). We let \( m = -1 \) if there is no preperiod.

There exists an \( i, 1 \leq i \leq 3 \), such that

\[
\sigma = [c_0, \ldots, c_r, b_{n-i}, \ldots, b_1, b_n, b_{n-1}, \ldots, b_{n-i+1}].
\]

Also \( i = 3 \) implies \( b_{n-1} = 1 \). When \( b_n = b_{n-1} = 1 \), then \( i = 3 \) if and only if \( m \geq 0 \) and \( r \geq 0 \).

Proofs were not given by Pavone. We list all cases for (1) and give proofs. We need two lemmas.

**Lemma 2.** If \( \xi = [a_0, a_1, \ldots] \), then

\[
-\xi = \begin{cases} 
[-a_0 - 1, 1, a_1 - 1, a_2, \ldots] & \text{if } a_1 > 1; \\
[-a_0 - 1, a_2 + 1, a_3, \ldots] & \text{if } a_1 = 1.
\end{cases}
\]
Remark 3. This is Lemma 3.1 of [1].

Lemma 4. Let $\rho = \frac{-b + \sqrt{d}}{2a} = [a_0, \ldots, a_m, b_1, \ldots, b_n]$ and $\bar{\rho} = \frac{-b - \sqrt{d}}{2a}$.

Then

$$\bar{\rho} = \begin{cases} 
[a_0, \ldots, a_m, -1, 1, b_n - 1, b_{n-1}, b_{n-2}, \ldots, b_1, b_n] & \text{if } b_n > 1; \\
[a_0, \ldots, a_m, -1, b_{n-1} + 1, b_{n-2}, b_{n-3}, \ldots, b_1, b_n, b_{n-1}] & \text{if } b_n = 1.
\end{cases}$$

The preperiod $a_0, \ldots, a_m$ can be absent.

Proof. We have $\rho = [a_0, \ldots, a_m, \theta]$, where $\theta = [b_1, \ldots, b_n]$. Taking conjugates gives

$$\bar{\rho} = [a_0, \ldots, a_m, \bar{\theta}] = [a_0, \ldots, a_m, -[0, b_n, \ldots, b_1]].$$

The desired conclusion now follows from Lemma 2. \qed

Our problem is to get rid of a negative partial quotient in the equations of Lemma 4. We use a matrix approach.

1. We first assume $b_n > 1$ and consider the identity

$$\begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_n - 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b_n - a_m & 1 \\ -1 & 0 \end{pmatrix}.$$  

Case (a). Assume $m \geq 1$ and $b_n - a_m \geq 1$. We use the matrix product

$$\begin{pmatrix} a_{m-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_n - a_m & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{m-1} (b_n - a_m) - 1 & a_{m-1} \\ b_n - a_m & 1 \end{pmatrix}.$$  

$$= \begin{pmatrix} a_{m-1} - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_n - a_m - 1 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Hence if $b_n > 1, m \geq 2$ and $b_n - a_m \geq 1$, we have

$$\sigma = \begin{cases} 
[a_0, \ldots, a_{m-2}, a_{m-1} - 1, 1, b_n - a_m - 1, b_{n-1}, \ldots, b_1, b_n] & \text{if } b_n - a_m > 1; \\
[a_0, \ldots, a_{m-2}, a_{m-1} - 1, 1 + b_{n-1}, b_{n-2}, \ldots, b_1, b_n, b_{n-1}] & \text{if } b_n - a_m = 1.
\end{cases}$$
while if \( b_n > 1, m = 1 \) and \( b_n - a_m \geq 1 \), we have

\[
\sigma = \begin{cases} 
[a_{m-1} - 1, 1, b_n - a_m - 1, b_{n-1}, \ldots, b_1, b_n] & \text{if } b_n - a_m > 1; \\
[a_{m-1} - 1, 1 + b_{n-1}, b_{n-2}, \ldots, b_1, b_n, b_{n-1}] & \text{if } b_n - a_m = 1.
\end{cases}
\]

On removing a zero partial quotient if \( a_{m-1} = 1 \), we obtain a continued fraction expansion for \( \sigma \) of the form \([1]\), where \( i = 1 \) if \( b_n - a_m > 1 \), and \( i = 2 \) if \( b_n - a_m = 1 \).

Case (b). Assume \( m \geq 1 \) and \( b_n - a_m = -b < 0 \), or \( m = 0 \). Let \( c = b_{n-1} \). Then

\[
\begin{pmatrix}
  b_n - a_m & 1 \\
  -1 & 0
\end{pmatrix}
\begin{pmatrix}
  b_{n-1} & 1 \\
  1 & 0
\end{pmatrix}
= \begin{pmatrix}
  -b & 1 \\
  -1 & 0
\end{pmatrix}
\begin{pmatrix}
  c & 1 \\
  1 & 0
\end{pmatrix}
= \begin{pmatrix}
  -bc + 1 & -b \\
  -c & -1
\end{pmatrix}
= -\begin{pmatrix}
  b - 1 & 1 \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  c - 1 & 1 \\
  1 & 0
\end{pmatrix}.
\]

Hence if \( b_n > 1, m \geq 1 \) and \( b_n - a_m < 0 \), we have

\[
\sigma = \begin{cases} 
[a_0, \ldots, a_{m-1}, a_m - b_n - 1, 1, b_{n-1} - 1, b_{n-2}, \ldots, b_1, b_n, b_{n-1}] & \text{if } b_{n-1} > 1; \\
[a_0, \ldots, a_{m-1}, a_m - b_n - 1, b_{n-2} + 1, b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}] & \text{if } b_{n-1} = 1.
\end{cases}
\]

while if \( b_n > 1 \) and \( m = 0 \), we have

\[
\sigma = \begin{cases} 
[a_0 - b_n - 1, 1, b_{n-1} - 1, b_{n-2}, \ldots, b_1, b_n, b_{n-1}] & \text{if } b_{n-1} > 1; \\
[a_0 - b_n - 1, b_{n-2} + 1, b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}] & \text{if } b_{n-1} = 1.
\end{cases}
\]

On removing a zero partial quotient if \( a_m - b_n = 1 \) and \( m \geq 1 \), we obtain a continued fraction expansion for \( \sigma \) which has the form \([1]\), where \( i = 2 \) if \( b_{n-1} > 1 \) and \( i = 3 \) if \( b_{n-1} = 1 \).

2. We now assume \( b_n = 1 \). Then

\[
\begin{pmatrix}
  a_m & 1 \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  -1 & 1 \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  b_{n-1} + 1 & 1 \\
  1 & 0
\end{pmatrix}
= -\begin{pmatrix}
  b_{n-1}(a_m - 1) & a_m - 1 \\
  b_{n-1} & 1
\end{pmatrix}
\begin{pmatrix}
  a_m - 2 & 1 \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  b_{n-1} - 1 & 1 \\
  1 & 0
\end{pmatrix}.
\]
Note that \( a_m \geq 2 \) if \( m \geq 1 \). Hence if \( b_n = 1 \) and \( m > 1 \), we get the continued fraction
\[
\sigma = \begin{cases} 
[a_0, \ldots, a_{m-1}, a_m - 2, 1, \overline{b_{n-1} - 1, b_n, \ldots, b_1, b_2} & \text{if } b_n - 1 > 1; \\
[a_0, \ldots, a_{m-1}, a_m - 2, 1 + b_{n-2}, \overline{b_{n-3}, \ldots, b_1, b_2, b_n, b_n - 2} & \text{if } b_n - 1 = 1.
\end{cases}
\]
while if \( b_n = 1 \) and \( m = 0 \), we get the continued fraction
\[
\sigma = \begin{cases} 
[a_0 - 2, 1, b_{n-1} - 1, \overline{b_n - 2, \ldots, b_1, b_2} & \text{if } b_{n-1} > 1; \\
[a_0 - 2, 1 + b_{n-2}, \overline{b_n - 3, \ldots, b_1, b_2, b_n, b_n - 2} & \text{if } b_{n-1} = 1.
\end{cases}
\]
On removing a zero partial quotient if \( a_m = 2 \), we obtain the continued fraction expansion for \( \sigma \) in the form (1), with \( i = 2 \) if \( b_{n-1} > 1 \), and \( i = 3 \) if \( b_{n-1} = 1 \).

3. Finally, we consider the case \( \rho = [b_1, \ldots, b_n] \). Then by Lemma 4
\[
\sigma = \begin{cases} 
[-1, 1, b_n - 1, \overline{b_{n-1}, \ldots, b_1, b_2} & \text{if } b_n > 1; \\
[-1, b_{n-1} + 1, \overline{b_n - 2, \ldots, b_1, b_2, b_n} & \text{if } b_n = 1.
\end{cases}
\]
Table I gives an expanded summary of all cases. There is similar table at [2] which assisted in the fine–tuning of Table I.

References

<table>
<thead>
<tr>
<th>( r )</th>
<th>Cases</th>
<th>Continued fraction expansion of ( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A11</td>
<td>( m+1 )</td>
<td>( b_n &gt; 1, m \geq 2, b_n - a_m &gt; 1, a_{m-1} &gt; 1 ) [ a_0, \ldots, a_{m-2}, a_{m-1} - 1, 1, b_n - a_m - 1, b_{n-1} - 1, b_n ]</td>
</tr>
<tr>
<td>A12</td>
<td>( m-1 )</td>
<td>( b_n &gt; 1, m \geq 2, b_n - a_m &gt; 1, a_{m-1} = 1 ) [ a_0, \ldots, a_{m-3}, a_{m-2} + 1, b_n - a_m - 1, b_{n-1} - 1, b_n ]</td>
</tr>
<tr>
<td>A13</td>
<td>1</td>
<td>( b_n &gt; 1, m = 2, b_n - a_2 &gt; 1, a_1 = 1 ) [ a_0 + 1, b_n - a_2 - 1, b_{n-1} - 1, 1, b_n ]</td>
</tr>
<tr>
<td>A21</td>
<td>( m )</td>
<td>( b_n &gt; 1, m \geq 2, b_n - a_m = 1, a_{m-1} &gt; 1 ) [ a_0, \ldots, a_{m-2}, a_{m-1} - 1, 1, b_{n-1} - 1, b_n - 1 ]</td>
</tr>
<tr>
<td>A22</td>
<td>( m-2 )</td>
<td>( b_n &gt; 1, m \geq 2, b_n - a_m = 1, a_{m-1} = 1 ) [ a_0, \ldots, a_{m-3}, a_{m-2} + 1 + b_{n-1}, b_{n-2} - 1, b_1, b_n, b_{n-1} ]</td>
</tr>
</tbody>
</table>
| A23   | 0      | \( b_n > 1, m = 2, b_n - a_2 = 1, a_1 = 1, a_0 \neq -1 \) \[ a_0 + 1 + b_{n-1}, b_{n-2} \ldots, b_1, b_n, b_{n-1} \]
| A24   | \(-1\) | \( b_n > 1, m = 2, b_n - a_2 = 1, a_1 = 1, a_0 = -1 \) \([b_{n-1} \ldots, b_1, b_n] \) |
| A25   | \( m+2 \) | \( b_n > 1, m \geq 1, a_m - b_1 > 1, b_{n-1} > 1 \) \[ a_0, \ldots, a_{m-1}, a_m - b_n - 1, 1, b_{n-1} - 1, b_n, b_{n-1} - 1 \] |
| A251  | \( m \) | \( b_n > 1, m \geq 2, a_m - b_1 = 1, b_{n-1} > 1 \) \[ a_0, \ldots, a_{m-2}, a_{m-1} + 1, b_{n-2} - 1, b_{n-1} - 1, b_1, b_n, b_{n-1} - 1 \] |
| A252  | 1      | \( b_n > 1, m = 1, a_m - b_1 = 1, b_{n-1} > 1 \) \[ a_0 + 1, b_{n-1} - 1, b_n, b_{n-1} - 1 \] |
| A26   | \( m+1 \) | \( b_n > 1, m \geq 1, a_m - b_1 > 1, b_{n-1} = 1 \) \[ a_0, \ldots, a_{m-1}, a_m - b_n - 1, 1, b_{n-2} + 1, b_{n-3} - 1, b_1, b_n, b_{n-1} - 2 \] |
| A261  | \( m-1 \) | \( b_n > 1, m \geq 2, a_m - b_1 = 1, b_{n-1} = 1 \) \[ a_0, \ldots, a_{m-2}, a_{m-1} + b_{n-2} + 1, b_{n-3} - 1, b_1, b_n, b_{n-1} - 2 \] |
| A262  | 0      | \( b_n > 1, m = 1, a_m - b_1 = 1, b_{n-1} = 1 \) \[ a_0 + b_{n-2} + 1, b_{n-3} - 1, b_1, b_n - 1, b_{n-2} \] |
| B1    | 2      | \( b_n > 1, m = 1, b_n - a_1 > 1 \) \[ a_0 - 1, 1, b_n - a_1 - 1, b_n, b_{n-1} \] |
| B2    | 1      | \( b_n > 1, m = 1, b_n - a_1 = 1 \) \[ -b_n + a_0 - 1, 1, b_n - a_1 - 1, b_n, b_{n-1} \] |
| B21   | 2      | \( b_n > 1, m = 0, b_n - a_1 > 1 \) \[ -b_n + a_0 - 1, 1, b_n - a_2 - 1, b_n, b_{n-1} \] |
| B22   | 1      | \( b_n > 1, m = 0, b_n - a_1 = 1 \) \[ -b_n + a_0 - 1, 1, b_n - a_2 = 1, b_n, b_{n-1} \] |
| C11   | \( m+2 \) | \( b_n = 1, b_{n-1} > 1, m \geq 1, a_m > 2 \) \[ a_0, \ldots, a_{m-2}, a_{m-1} - 2, b_{n-2} - 1, b_{n-1} - 1, b_n - 1, b_{n-1} - 1 \] |
| C12   | \( m+1 \) | \( b_n = 1, b_{n-1} \geq 1, m \geq 1, a_m > 2 \) \[ a_0, \ldots, a_{m-2}, a_{m-1} - 2, b_{n-2} + 1, b_{n-3} - 1, b_1, b_n, b_{n-1} - 2 \] |
| C21   | \( m \) | \( b_n = 1, b_{n-1} \geq 1, m \geq 2, a_m = 2 \) \[ a_0, \ldots, a_{m-2}, a_{m-1} - 1, b_{n-2} + 1, b_{n-3} - 1, b_1, b_n, b_{n-1} - 2 \] |
| C22   | \( m-1 \) | \( b_n = 1, b_{n-1} \geq 1, m \geq 2, a_m = 2 \) \[ a_0, \ldots, a_{m-2}, a_{m-1} - 1, b_{n-2} + 1, b_{n-3} - 1, b_1, b_n, b_{n-1} - 2 \] |
| C31   | 1      | \( b_n = 1, b_{n-1} > 1, m = 1, a_1 = 2 \) \[ a_0 + b_{n-2} + 1, b_{n-3} - 1, b_1, b_n, b_{n-1} \] |
| C32   | 0      | \( b_n = 1, b_{n-1} = 1, m = 1, a_1 = 2, a_0 \neq -1 \) \[ a_0 + b_{n-2} + 1, b_{n-3} - 1, b_1, b_n, b_{n-1} \] |
| C33   | \(-1\) | \( b_n = 1, b_{n-1} = 1, m = 1, a_1 = 2, a_0 = -1 \) \[ b_{n-2} - 1, b_1, b_n, b_{n-1} \] |
| D1    | 2      | \( b_n = 1, b_{n-1} > 1, m = 0 \) \[ a_0 = b_n - 1, 1, b_{n-1} - 1, b_{n-2} - 1, b_1, b_n, b_{n-1} \] |
| D2    | 1      | \( b_n = 1, b_{n-1} = 1, m = 0 \) \[ a_0 = b_n - 1, b_{n-2} + 1, b_{n-3} - 1, b_1, b_n, b_{n-1} - 2 \] |
| G1    | 2      | \( b_n > 1, m = -1 \) \[ -1, 1, b_n - 1, b_{n-1} - 1, b_1, b_n, b_{n-1} \] |
| G2    | 1      | \( b_n = 1, m = -1 \) \[ -1, b_{n-1} + 1, b_{n-2} - 1, b_1, b_n, b_{n-1} \] |