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A Generalization of the Syracuse Algorithm in $F_q[x]$

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In this note we remark that while much of the theory of a recent paper of Matthews and Watts on mappings $T: \mathbf{Z} \to \mathbf{Z}$ generalizing the Syracuse algorithm also goes over to mappings $T: F_q[x] \to F_q[x]$, the conjectural picture is not as clear for polynomials. We exhibit two divergent trajectories which possess an unexpected regularity, and which do not obey a certain expected uniformity of distribution. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let $F_q[x]$ be the ring of polynomials over F_q, a field with q elements. Let $d \in F_q[x]$, $t = \deg d > 0$ and let R_d be a complete set of residues mod d. Then $R_d = \{x_1, ..., x_{N(d)}\}$, where $N(d) = q^{\deg d}$. For $i = 1, ..., N(d)$, let $m_i \in F_q[x]$, $\gcd(m_i, d) = 1$. Also let $r_i \in R_d$ be defined by $r_i \equiv m_i x_i (\text{mod } d)$. Then we can define a mapping $T: F_q[x] \to F_q[x]$ by

$$T(f) = \frac{m_i f - r_i}{d} \qquad \text{if } f \equiv x_i (\text{mod } d). \tag{1.1}$$

This mapping is the analogue for $F_q[x]$ of a mapping $T: \mathbf{Z} \to \mathbf{Z}$ studied by Matthews and Watts [1]. As in [1] we are interested in the distribution mod d^α of sequences of iterates $T^K(f)$, $K \geqq 0$, $f \in F_q[x]$, where the sequence is not eventually periodic. (We call these sequences divergent trajectories.)

As in [1], T extends to a continuous mapping, $T: G \to G$, where G is the d-adic completion of $F_q[x]$; also T is measure-preserving and strongly mixing with respect to the Haar measure μ on G which satisfies $\mu(B(j, d^\alpha)) = 1/N(d^\alpha)$, where $B(j, d^\alpha) = \{f \in F_q[x] \mid f \equiv j (\text{mod } d^\alpha)\}$.

It is natural to suggest that analogs of Conjectures (i–iv) of [1] exist. However the situation appears to be more complicated and harder to predict here. It is the purpose of this note to give examples of the failure of

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Conjecture (iv); i.e., we will produce divergent trajectories $\{T^K(f)\}_{K \geqslant 0}$ for which

$$\lim_{N \to \infty} \frac{1}{N} \operatorname{card}\{K \leqq N \mid T^K(f) \equiv j (\bmod d)\} \tag{1.2}$$

does not exist.

In the second example it seems fairly certain that most trajectories are eventually periodic. We show only that there are infinitely many periods.

2. The Examples

We need the following result about certain d-adically convergent series in G.

Lemma 2.1. *Let $f \in G$ and suppose that*

$$f = \sum_{K=0}^{\infty} \frac{r_{i_K} d^K}{m_{i_0} \cdots m_{i_K}}, \tag{2.1}$$

where i_K, $K \geqq 0$, is a sequence of integers satisfying $1 \leqq i_K \leqq N(d)$. Then for $s \geqq 0$,

$$T^s(f) = \sum_{K=s}^{\infty} \frac{r_{i_K} d^{K-s}}{m_{i_s} \cdots m_{i_K}} \tag{2.2}$$

and hence

$$T^s(f) \equiv x_{i_s} (\bmod d) \qquad \text{if } s \geqq 0. \tag{2.3}$$

Proof. (2.2) follows from induction.

Then (2.3) follows from the congruence

$$T^s(f) \equiv r_{i_s}/m_{i_s} \equiv x_{i_s} (\bmod d).$$

Example 1. Let $T: F_2[x] \to F_2[x]$ be defined by

$$T(f) = \begin{cases} \dfrac{f}{x} & \text{if } f \equiv 0 \ (\bmod x), \\[2ex] \dfrac{(x+1)^3 f + 1}{x} & \text{if } f \equiv 1 \ (\bmod x). \end{cases} \tag{2.4}$$

(Here $d = x$, $R_x = \{x_1, x_2\}$, where $x_1 = 0$, $x_2 = 1$, $m_1 = 1$, $m_2 = (x+1)^3$, $r_1 = 0$, $r_2 = 1$.)

We prove that the trajectory $\{T^K(1)\}_{K \geqq 0}$ is divergent by showing that if $K_n = 6(2^n - 1)$, $n \geqq 0$, then

$$T^{K_n}(1) = (1 + x^2 + x^3)^{2^n - 1}. \tag{2.5}$$

We also prove that if i_K is defined for $K \geqq 0$ by

$$i_K = \begin{cases} 2 & \text{if } K_n \leqq K < K_n + 2^{n+1}, \\ 1 & \text{if } K_n + 2^{n+1} \leqq K < K_n + 3 \cdot 2^n, \\ 2 & \text{if } K_n + 3 \cdot 2^n \leqq K < K_n + 2^{n+2}, \\ 1 & \text{if } K_n + 2^{n+2} \leqq K < K_{n+1}, \end{cases} \tag{2.6}$$

then

$$T^s(1) \equiv x_{i_s} (\text{mod } x) \qquad \text{for } s \geqq 0. \tag{2.7}$$

Remarks. 1. It is then easy to verify that

$$\begin{aligned} &\text{card}\{K < K_n \mid T^K(1) \equiv 0 \ (\text{mod } x)\} \\ &\quad = \text{card}\{K < K_n \mid T^K(1) \equiv 1 \ (\text{mod } x)\} = K_n/2 \end{aligned} \tag{2.8}$$

and that

$$\text{card}\{K < 2^{n+3} - 6 \mid T^K(1) \equiv 0 \ (\text{mod } x)\} = 3 \cdot 2^n - 3 \tag{2.9}$$

and

$$\text{card}\{K < 2^{n+3} - 6 \mid T^K(1) \equiv 1 \ (\text{mod } x)\} = 5 \cdot 2^n - 3. \tag{2.10}$$

Consequently the limit (1.2) does not exist for $j = 0$ or 1.

2. Most trajectories seem to be divergent, though not necessarily possessing the above regularity exhibited by (2.7).

Proof. Let $f \in G$ be defined by (2.1), where i_K, $K \geqq 0$, is defined by (2.6). Also let

$$f_n = (1 + x^2 + x^3)^{2^n - 1} \qquad \text{for } n \geqq 0. \tag{2.11}$$

Then if $S_n \in G$ is defined by

$$S_n = \sum_{K = K_n}^{K_{n+1} - 1} \frac{r_{i_K} x^{K - K_n}}{m_{i_{K_n}} \cdots m_{i_K}}, \tag{2.12}$$

we easily verify that

$$S_n = \left(1 + \left(\frac{x}{p}\right)^{2^{n+1}} + \frac{x^{3 \cdot 2^n}}{p^{2^{n+1}}} + \frac{x^{2^{n+2}}}{p^{3 \cdot 2^n}}\right) \Big/ (1 + x^2 + x^3), \tag{2.13}$$

where $p = (1 + x)^3$.

We then verify that the f_n satisfy

$$f_n = S_n + \frac{x^{3 \cdot 2^{n+1}}}{p^{3 \cdot 2^n}} f_{n+1}. \tag{2.14}$$

(The proof is straightforward and is omitted.)

Repeated use of (2.14) gives

$$\begin{aligned} f_n &= \sum_{i=0}^{\infty} \frac{x^{2^n K_i}}{p^{2^{n-1} K_i}} S_{n+i} = \sum_{K=K_n}^{\infty} \frac{r_{i_K} x^{K-K_n}}{m_{i_{K_n}} \cdots m_{i_K}} \\ &= T^{K_n}(f) \qquad \text{by (2.2),} \end{aligned} \tag{2.15}$$

thereby proving (2.5), since it now follows that $f_0 = f$, and from (2.11) we also have $f_0 = 1$. Then (2.7) follows from (2.3).

EXAMPLE 2. Let $T: F_2[x] \to F_2[x]$ be defined by

$$T(f) = \begin{cases} \dfrac{f}{x} & \text{if } f \equiv 0 \pmod{x}, \\ \dfrac{(x+1)^2 f + 1}{x} & \text{if } f \equiv 1 \pmod{x}. \end{cases} \tag{2.16}$$

We can similarly prove that the trajectory $\{T^K(1+x+x^3)\}_{K \geqq 0}$ is divergent by showing that if $L_n = 5(2^n - 1)$, $n \geqq 0$, then

$$T^{L_n}(1+x+x^3) = \frac{1 + x^{3 \cdot 2^n + 1} + x^{3 \cdot 2^n + 2}}{1 + x + x^2}. \tag{2.17}$$

Also

$$T^K(1+x+x^3) \equiv 1 \pmod{x} \qquad \text{if } L_n \leqq K < L_n + 3 \cdot 2^n, \tag{2.18}$$

while if $L_n + 3 \cdot 2^n \leqq K < L_{n+1}$ then

$$T^K(1+x+x^3) \equiv 1 \pmod{x} \Leftrightarrow K \equiv 1 \pmod{2}. \tag{2.19}$$

Again the limits (1.2) do not exist when $j = 0$ or 1.

Finally let

$$g_n = 1 + x + \cdots + x^{2^n - 2} = \frac{1 + x^{2^n - 1}}{1 + x} \qquad \text{for } n \geqslant 1. \tag{2.20}$$

We prove that the trajectory $\{T^K(g_n)\}_{K \geqslant 0}$ is periodic by showing that

$$T^{2^n}(g_n) = g_n. \tag{2.21}$$

We also prove that

$$T^s(g_n) \equiv \begin{cases} 1 \pmod{x} & \text{if } 0 < s < 2^n - 1,\ s \text{ odd, or } s = 0; \\ 0 \pmod{x} & \text{if } 0 < s < 2^n,\ s \text{ even, or } s = 2^n - 1. \end{cases} \tag{2.22}$$

Proof. Using the notation of Lemma 2.1, let $r_1 = 0$, $r_2 = 1$, $x_1 = 0$, $x_2 = 1$. Also let

$$i_j = \begin{cases} 2 & \text{if } 0 < j < 2^n - 1,\ j \text{ odd, or } j = 0; \\ 1 & \text{if } 0 < j < 2^n,\ j \text{ even, or } j = 2^n - 1, \end{cases} \tag{2.23}$$

$$i_{j+2^n} = i_j \qquad \text{for } j \geqslant 0. \tag{2.24}$$

Then if $f \in G$ is defined by (2.1), it follows from (2.2) and (2.24) that $T^{2^n}(f) = f$. Also (2.22) follows from (2.3) and (2.23). It remains to prove that $f = g_n$. If $n \geqslant 2$, we have from (2.1) that

$$f = T_n + \frac{x^{2^n}}{q^{2^{n-1}}} T_n + \left(\frac{x^{2^n}}{q^{2^{n-1}}}\right)^2 T_n + \cdots, \tag{2.25}$$

where

$$q = x^2 + 1 \qquad \text{and} \qquad T_n = \frac{1}{q} + \frac{x}{q^2} + \frac{x^3}{q^3} + \cdots + \frac{x^{2^n-3}}{q^{2^{n-1}}}.$$

Hence

$$f_n = T_n \bigg/ \left(1 + \frac{x^{2^n}}{q^{2^{n-1}}}\right) = q^{2^{n-1}} T_n,$$

which easily reduces to g_n.

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Reference

1. K. R. Matthews and A. M. Watts, A generalization of Hasse's generalization of the Syracuse algorithm, *Acta Arith.* **43** (1984), 75–83.