A Generalization of the Syracuse Algorithm in $F_q[x]$

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In this note we remark that while much of the theory of a recent paper of Matthews and Watts on mappings $T: \mathbb{Z} \rightarrow \mathbb{Z}$ generalizing the Syracuse algorithm also goes over to mappings $T: F_q[x] \rightarrow F_q[x]$, the conjectural picture is not as clear for polynomials. We exhibit two divergent trajectories which possess an unexpected regularity, and which do not obey a certain expected uniformity of distribution.

1. INTRODUCTION

Let $F_q[x]$ be the ring of polynomials over $F_q$, a field with $q$ elements. Let $d \in F_q[x]$, $t = \deg d > 0$ and let $R_d$ be a complete set of residues mod $d$. Then $R_d = \{x_1, \ldots, x_{N(d)}\}$, where $N(d) = q^t$. For $i = 1, \ldots, N(d)$, let $m_i \in F_q[x]$, $\gcd(m_i, d) = 1$. Also let $r_i \equiv x_i (\text{mod } d)$. Then we can define a mapping $T: F_q[x] \rightarrow F_q[x]$ by

$$T(f) = \frac{m_i f - r_i}{d} \quad \text{if } f \equiv x_i (\text{mod } d). \quad (1.1)$$

This mapping is the analogue for $F_q[x]$ of a mapping $T: \mathbb{Z} \rightarrow \mathbb{Z}$ studied by Matthews and Watts [1]. As in [1] we are interested in the distribution mod $d^n$ of sequences of iterates $T^k(f)$, $K \geq 0, f \in F_q[x]$, where the sequence is not eventually periodic. (We call these sequences divergent trajectories.)

As in [1], $T$ extends to a continuous mapping, $T: G \rightarrow G$, where $G$ is the $d$-adic completion of $F_q[x]$; also $T$ is measure-preserving and strongly mixing with respect to the Haar measure $\mu$ on $G$ which satisfies $\mu(B(j, d^n)) = 1/N(d^n)$, where $B(j, d^n) = \{f \in F_q[x] \mid f \equiv j (\text{mod } d^n)\}$.

It is natural to suggest that analogs of Conjectures (i–iv) of [1] exist. However the situation appears to be more complicated and harder to predict here. It is the purpose of this note to give examples of the failure of
Conjecture (iv); i.e., we will produce divergent trajectories \(\{T^K(f)\}_{K \geq 0}\) for which

\[
\lim_{N \to \infty} \frac{1}{N} \text{card}\{K \leq N | T^K(f) \equiv j (\text{mod } d)\} = 0
\]

(1.2)

does not exist.

In the second example it seems fairly certain that most trajectories are eventually periodic. We show only that there are infinitely many periods.

2. THE EXAMPLES

We need the following result about certain \(d\)-adically convergent series in \(G\).

**Lemma 2.1.** Let \(f \in G\) and suppose that

\[
f = \sum_{K=0}^{\infty} \frac{r_K d^K}{m_0 \cdots m_K},
\]

(2.1)

where \(i_K, K \geq 0\), is a sequence of integers satisfying \(1 \leq i_K \leq N(d)\). Then for \(s \geq 0\),

\[
T^s(f) = \sum_{K=s}^{\infty} \frac{r_K d^{K-s}}{m_i \cdots m_{i_K}}
\]

(2.2)

and hence

\[
T^s(f) \equiv x_i (\text{mod } d) \quad \text{if } s \geq 0.
\]

(2.3)

**Proof:** (2.2) follows from induction.

Then (2.3) follows from the congruence

\[
T^s(f) \equiv r_i/m_i = x_i (\text{mod } d).
\]

**Example 1.** Let \(T: F_2[x] \to F_2[x]\) be defined by

\[
T(f) = \begin{cases} 
\frac{f}{x} & \text{if } f \equiv 0 \pmod{x}, \\
\frac{(x+1)^3 f + 1}{x} & \text{if } f \equiv 1 \pmod{x}.
\end{cases}
\]

(2.4)

(Here \(d = x\), \(R_x = \{x_1, x_2\}\), where \(x_1 = 0, x_2 = 1, m_1 = 1, m_2 = (x+1)^3, r_1 = 0, r_2 = 1\).)
We prove that the trajectory \( \{ T^K(1) \}_{K \geq 0} \) is divergent by showing that if 
\( K_n = 6(2^n - 1), \ n \geq 0 \), then
\[
T^K_n(1) = (1 + x^2 + x^3)^{2^n - 1}. \tag{2.5}
\]

We also prove that if \( i_K \) is defined for \( K \geq 0 \) by
\[
i_K = \begin{cases} 
2 & \text{if } K_n \leq K < K_n + 2^n + 1, \\
1 & \text{if } K_n + 2^n + 1 \leq K < K_n + 3 \cdot 2^n, \\
2 & \text{if } K_n + 3 \cdot 2^n \leq K < K_n + 2^{n+2}, \\
1 & \text{if } K_n + 2^{n+2} \leq K < K_{n+1},
\end{cases} \tag{2.6}
\]
then
\[
T^n(1) \equiv x_{i_K}(\text{mod } x) \quad \text{for } s \geq 0. \tag{2.7}
\]

Remarks. 1. It is then easy to verify that
\[
\text{card} \{ K < K_n \mid T^K(1) \equiv 0 (\text{mod } x) \} = K_n/2 \tag{2.8}
\]
and that
\[
\text{card} \{ K < 2^n + 3 - 6 \mid T^K(1) \equiv 0 (\text{mod } x) \} = 3 \cdot 2^n - 3 \tag{2.9}
\]
and
\[
\text{card} \{ K < 2^n + 3 - 6 \mid T^K(1) \equiv 1 (\text{mod } x) \} = 5 \cdot 2^n - 3. \tag{2.10}
\]
Consequently the limit (1.2) does not exist for \( j = 0 \) or 1.

2. Most trajectories seem to be divergent, though not necessarily possessing the above regularity exhibited by (2.7).

Proof. Let \( f \in G \) be defined by (2.1), where \( i_K, K \geq 0 \), is defined by (2.6). Also let
\[
f_n = (1 + x^2 + x^3)^{2^n - 1} \quad \text{for } n \geq 0. \tag{2.11}
\]
Then if \( S_n \in G \) is defined by
\[
S_n = \sum_{K = K_n}^{K_n+1-1} \frac{r_{ik} x^{K-K_n}}{m_{ik} \cdots m_{ik}}, \tag{2.12}
\]
we easily verify that
\[
S_n = \left( 1 + \frac{x^2}{p} + \frac{x^3}{p^{2^{n+1}}} + \frac{x^{2n+3}}{p^{2^{n+2}}} \right) \left( 1 + x^2 + x^3 \right), \tag{2.13}
\]
where \( p = (1 + x)^3 \).
We then verify that the $f_n$ satisfy
\[ f_n = S_n + \frac{x^{3 \cdot 2^n + 1}}{p^{3 \cdot 2^n}} f_{n+1}. \] (2.14)

(The proof is straightforward and is omitted.)

Repeated use of (2.14) gives
\[ f_n = \sum_{i=0}^{\infty} \frac{x^{2^n \cdot 2^i}}{p^{2^n - i}}, \quad S_n = \sum_{k=K_0}^{\infty} \frac{a_k x^{k - K_0}}{m_{i_k} \cdots m_i} \]
\[ = T^{K_0}(f) \quad \text{by (2.2),} \]

thereby proving (2.5), since it now follows that $f_0 = f$, and from (2.11) we also have $f_0 = 1$. Then (2.7) follows from (2.3).

**Example 2.** Let $T: F_2[x] \rightarrow F_2[x]$ be defined by
\[ T(f) = \begin{cases} \frac{f}{x} & \text{if } f \equiv 0 \pmod{x}, \\ \frac{(x + 1)^2 f + 1}{x} & \text{if } f \equiv 1 \pmod{x}. \end{cases} \] (2.16)

We can similarly prove that the trajectory \( \{T^k(1 + x + x^3)\}_{K \geq 0} \) is divergent by showing that if $L_n = 5(2^n - 1)$, $n \geq 0$, then
\[ T^{L_n}(1 + x + x^3) = \frac{1 + x^{3 \cdot 2^n + 1} + x^{3 \cdot 2^n + 2}}{1 + x + x^2}. \] (2.17)

Also
\[ T^K(1 + x + x^3) \equiv 1 \pmod{x} \quad \text{if } L_n \leq K < L_n + 3 \cdot 2^n, \] (2.18)

while if $L_n + 3 \cdot 2^n \leq K < L_{n+1}$ then
\[ T^K(1 + x + x^3) \equiv 1 \pmod{x} \iff K \equiv 1 \pmod{2}. \] (2.19)

Again the limits (1.2) do not exist when $j = 0$ or 1.

Finally let
\[ g_n = 1 + x + \cdots + x^{2^n - 2} = \frac{1 + x^{2^n - 1}}{1 + x} \quad \text{for } n \geq 1. \] (2.20)

We prove that the trajectory \( \{T^K(g_n)\}_{K \geq 0} \) is periodic by showing that
\[ T^{2^n}(g_n) = g_n. \] (2.21)
We also prove that
\[ T^r(g_n) = \begin{cases} 1 \pmod{x} & \text{if } 0 < s < 2^n - 1, \text{ s odd, or } s = 0; \\ 0 \pmod{x} & \text{if } 0 < s < 2^n, \text{ s even, or } s = 2^n - 1. \end{cases} \] (2.22)

**Proof.** Using the notation of Lemma 2.1, let \( r_1 = 0, \; r_2 = 1, \; x_1 = 0, \; x_2 = 1 \). Also let
\[ i_j = \begin{cases} 2 & \text{if } 0 < j < 2^n - 1, \text{ j odd, or } j = 0; \\ 1 & \text{if } 0 < j < 2^n, \text{ j even, or } j = 2^n - 1, \end{cases} \] (2.23)
\[ i_j + 2^n = i_j \quad \text{for } j \geq 0. \] (2.24)

Then if \( f \in G \) is defined by (2.1), it follows from (2.2) and (2.24) that \( T^x(f) = f \). Also (2.22) follows from (2.3) and (2.23). It remains to prove that \( f = g_n \). If \( n \geq 2 \), we have from (2.1) that
\[ f = T_n \frac{x^{2^n}}{q^{2^n - 1}} T_n + \left( \frac{x^{2^n}}{q^{2^n - 1}} \right)^2 T_n + \cdots, \] (2.25)
where
\[ q = x^2 + 1 \quad \text{and} \quad T_n = \frac{1}{q} + \frac{x}{q^2} + \frac{x^2}{q^3} + \cdots + \frac{x^{2^n-3}}{q^{2^n-1}}. \]

Hence
\[ f_n = T_n \left( 1 + \frac{x^{2^n}}{q^{2^n - 1}} \right) = q^{2^n-1} T_n, \]
which easily reduces to \( g_n \).

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**Reference**