ON A BILINEAR FORM ASSOCIATED WITH THE LARGE SIEVE

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1. *Introduction*

Let $x_1, \ldots, x_R$ $(R \geqslant 2)$ be any real numbers satisfying

$$\|x_r - x_s\| \geqslant \delta > 0 \quad \text{if} \quad r \neq s, \tag{1}$$

where $\|\theta\|$ is the distance from θ to the nearest integer.

The author has recently proved the following result (see [1]).

Let γ be a constant for which the inequality

$$\left| \sum_{\substack{r=1 \\ r \neq s}}^{R} \sum_{s=1}^{R} \frac{u_r v_s}{\sin \pi(x_r - x_s)} \right| \leqslant \gamma \tag{2}$$

holds for all real numbers $u_1, \ldots, u_R, v_1, \ldots, v_R$ satisfying

$$\sum_r u_r^2 = \sum_s v_s^2 = 1.$$

Then if $a_{M+1}, \ldots, a_{M+N}$ are arbitrary complex numbers, we have the large sieve inequality

$$\sum_{r=1}^{R} \left| \sum_{n=M+1}^{M+N} a_n e(nx_r) \right|^2 \leqslant (N+\gamma) \sum_{n=M+1}^{M+N} |a_n|^2.$$

In the present paper we show that (2) holds with

$$\gamma = \left(3 + \frac{1}{2\pi} + \frac{\pi}{3}\right)\delta^{-1}, \quad \left(3 + \frac{1}{2\pi} + \frac{\pi}{3} = 4{\cdot}2\ldots\right).$$

Without loss of generality, we may assume that

$$0 \leqslant x_1 < \ldots < x_R < 1.$$

For on writing x_r as

$$x_r = [x_r] + \{x_r\},$$

a sum of integral and fractional parts respectively, we have

$$\sum_{r \neq s} \frac{u_r v_s}{\sin \pi(x_r - x_s)} = \sum_{r \neq s} \frac{u_r' v_s'}{\sin \pi(\{x_r\} - \{x_s\})},$$

where $u_r' = (-1)^{[x_r]} u_r$ and $v_s' = (-1)^{[x_s]} v_s$. Inequality (1) is now replaced by

$$\delta \leqslant x_r - x_s \leqslant 1 - \delta \quad \text{if} \quad 1 \leqslant s < r \leqslant R, \tag{3}$$

which may be strengthened to

$$(r-s) \leqslant x_r - x_s \leqslant 1 - (R - (r-s))\delta \quad \text{if} \quad 1 \leqslant s \leqslant r \leqslant R. \tag{4}$$

The special case $r = s$ gives

$$R\delta \leqslant 1. \tag{5}$$

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2. Write

$$\sum_{r\neq s}\frac{u_r v_s}{\sin\pi(x_r-x_s)}-\frac{1}{\pi}\sum_{r\neq s}\frac{u_r v_s}{x_r-x_s}=\sum_{r\neq s}a_{rs}u_r v_s. \tag{6}$$

We proceed to show that

$$\left|\sum_{r\neq s}a_{rs}u_r v_s\right|<\left(2+\frac{1}{2\pi}\right)\delta^{-1}. \tag{7}$$

Consider the odd function ψ defined for $0<|x|<1$ by

$$\frac{\pi}{\sin\pi x}=\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}+\psi(x). \tag{8}$$

It is well known (see [2; pp. 112–113]) that

$$\psi(x)=\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{n+x}-\frac{1}{n-x}\right).$$

Hence

$$\psi'(x)=\sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{(n+x)^2}+\frac{1}{(n-x)^2}\right)$$

and consequently $\psi'(x)<0$ if $0<|x|<1$.
Hence ψ is a decreasing function over $0<|x|<1$. In particular,

$$|\psi(x)|\leqslant\lim_{x\to-1}\psi(x)=\tfrac{1}{2}. \tag{9}$$

From (6) and (8) we have that

$$\pi\sum_{r\neq s}a_{rs}u_r v_s=\sum_{r\neq s}\frac{u_r v_s}{1-(x_r-x_s)}-\sum_{r\neq s}\frac{u_r v_s}{1+(x_r-x_s)}+\sum_{r\neq s}u_r v_s\psi(x_r-x_s)$$

$$=\sum_r\sum_s\frac{u_r v_s}{1-(x_r-x_s)}-\sum_r\sum_s\frac{u_r v_s}{1+(x_r-x_s)}+\sum_{r\neq s}u_r v_s\psi(x_r-x_s). \tag{10}$$

Now it is easy to verify from (4) and (5) that the relation

$$1-(x_r-x_s)\geqslant\big(R-(r-s)\big)\delta \tag{11}$$

holds for all r and s. Also

$$1+(x_r-x_s)\geqslant\big(R-(s-r)\big)\delta. \tag{12}$$

Inequalities (9), (11) and (12) together with (10) give

$$\pi\left|\sum_{r\neq s}a_{rs}u_r v_s\right|\leqslant\delta^{-1}\sum_r\sum_s\frac{|u_r v_s|}{R-r+s}+\delta^{-1}\sum_r\sum_s\frac{|u_r v_s|}{R-s+r}+\tfrac{1}{2}\sum_{r\neq s}|u_r v_s|$$

$$=\delta^{-1}\sum_{a=0}^{R-1}\sum_{b=0}^{R-1}\frac{|u_{R-a}v_{b+1}|}{a+b+1}+\delta^{-1}\sum_{a=0}^{R-1}\sum_{b=0}^{R-1}\frac{|u_{b+1}v_{R-a}|}{a+b+1}+\tfrac{1}{2}\sum_{r\neq s}|u_r v_s|. \tag{13}$$

The Hilbert inequality (see [3; Theorem 294, p. 212]) gives

$$\sum_{a=0}^{R-1}\sum_{b=0}^{R-1}\frac{|u_{R-a}v_{b+1}|}{a+b+1}\leqslant\pi\left(\sum_{a=0}^{R-1}u_{R-a}^2\right)^{\frac{1}{2}}\left(\sum_{b=0}^{R-1}v_{b+1}^2\right)^{\frac{1}{2}}=\pi. \tag{14}$$

Similarly,

$$\sum_{a=0}^{R-1}\sum_{b=0}^{R-1}\frac{|u_{b+1}v_{R-a}|}{a+b+1}\leqslant \pi. \tag{15}$$

Also,

$$\sum_{r\neq s}|u_r v_s| \leqslant (R-1)\left(\sum_{r=1}^{R}u_r^2\right)^{\frac{1}{2}}\left(\sum_{s=1}^{R}v_s^2\right)^{\frac{1}{2}} = R-1.$$

(See [3; Theorem 275, p.198].) Hence, from (5), we have that

$$\sum_{r\neq s}|u_r v_s| < \delta^{-1}. \tag{16}$$

Inequality (7) now follows from (13), (14), (15) and (16).

3. Let $S = [\delta^{-1}]$, so that

$$\delta^{-1} = S+\theta, \quad 0\leqslant \theta < 1.$$

Also

$$1 = S\delta+\theta\delta. \tag{17}$$

I_k ($k = 1, \ldots, S$) is defined to be the interval

$$[x_1+(k-1)\delta, \quad x_1+k\delta[.$$

These intervals are disjoint and cover the interval

$$[x_1, x_R],$$

since by (3) and (17) we have that

$$\begin{aligned} x_R-x_1 \leqslant 1-\delta &= S\delta+\theta\delta-\delta \\ &= \big(S-(1-\theta)\big)\delta < S\delta. \end{aligned}$$

Also each I_k contains at most one x_r. We now define integers $k_1, \ldots, k_R$ by the condition

$$x_r \in I_{k_r}.$$

Then

$$1 = k_1 < \ldots < k_R \leqslant S, \tag{18}$$

and

$$x_r = x_1+(k_r-1)\delta+\theta_r, \quad \text{where} \quad 0\leqslant \theta_r < \delta. \tag{19}$$

Writing

$$\frac{1}{\pi}\sum_{r\neq s}\frac{u_r v_s}{x_r-x_s} - \frac{\delta^{-1}}{\pi}\sum_{r\neq s}\frac{u_r v_s}{k_r-k_s} = \sum_{r\neq s} b_{rs}u_r v_s,$$

where

$$\pi b_{rs} = \frac{1}{x_r-x_s} - \frac{1}{(k_r-k_s)\delta},$$

we proceed to verify that

$$\left|\sum_{r\neq s} b_{rs}u_r v_s\right| < \frac{\pi}{3}\delta^{-1}. \tag{20}$$

We have

$$\pi b_{rs} = \frac{\theta_s-\theta_r}{(x_r-x_s)(k_r-k_s)\delta},$$

and hence

$$\pi|b_{rs}| < \frac{\delta^{-1}}{(r-s)^2},$$

by (4), (18) and (19). Hence

$$\pi\left|\sum_{r\neq s} b_{rs}u_r v_s\right| \leqslant \delta^{-1}\sum_{r\neq s}\frac{|u_r v_s|}{(r-s)^2} < \delta^{-1}\frac{\pi^2}{3}.$$

(See [3; Theorem 275, p. 198].)

4. Our proof is concluded by showing that

$$\left|\sum_{r\neq s}\frac{u_r v_s}{k_r-k_s}\right| \leqslant \pi. \tag{21}$$

We have

$$\sum_{r\neq s}\frac{u_r v_s}{k_r-k_s} = \sum_{\substack{k=1\\ k\neq j}}^{S}\sum_{j=1}^{S}\frac{u_k' v_j'}{k-j},$$

where

$$u_k' = \begin{cases} u_{k_r} & \text{if } k = k_r,\\ 0 & \text{otherwise.}\end{cases}$$

Similarly for v_j'.

The Hilbert inequality

$$\left|\sum_{\substack{k\\ k\neq j}}\sum_{j}\frac{u_k' v_j'}{k-j}\right| \leqslant \pi\left(\sum_k (u_k')^2\right)^{\frac{1}{2}}\left(\sum_j (v_j')^2\right)^{\frac{1}{2}},$$

(see [3; Theorem 294, p. 212]) yields (21), as

$$\sum_k (u_k')^2 = \sum_j (v_j')^2 = 1.$$

Finally, on combining inequalities (7), (20) and (21) we have

$$\left|\sum_{r\neq s}\frac{u_r v_s}{\sin\pi(x_r-x_s)}\right| \leqslant \left(3+\frac{1}{2\pi}+\frac{\pi}{3}\right)\delta^{-1}.$$

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References

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