ON AN INEQUALITY OF DAVENPORT AND HALBERSTAM

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1. *Introduction*

Let

$$S(x) = \sum_{n=M+1}^{M+N} a_n\, e(nx) \qquad (e(\theta) = e^{2\pi i\theta}),$$

where $a_{M+1}, \ldots, a_{M+N}$ are any complex numbers. Let $x_1, \ldots, x_R$ $(R \geqslant 2)$ be any real numbers satisfying

$$\|x_r - x_s\| \geqslant \delta > 0 \quad \text{for} \quad r \neq s,$$

where $\|\theta\|$ is the distance from θ to the nearest integer.

Inequalities of the form

$$\sum_{r=1}^{R} |S(x_r)|^2 \leqslant \kappa(N, \delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2$$

were first obtained by Davenport and Halberstam [**1**] with

$$\kappa(N, \delta^{-1}) = 2{\cdot}2 \max(N, \delta^{-1}).$$

Other estimates for $\kappa(N, \delta^{-1})$ are $\pi N + \delta^{-1}$ (Gallagher [**2**]), $2\max(N, \delta^{-1})$ (Ming-Chit Liu [**3**], Bombieri and Davenport [**4**]), $(N^{\frac{1}{2}} + \delta^{-\frac{1}{2}})^2$ (Bombieri and Davenport [**4**]), $N + 5\delta^{-1}$ (Bombieri and Davenport [**5**]).

(In the following, variables r and s range over $1, \ldots, R$, and variables m and n range over $M+1, \ldots, M+N$.)

The discussion here is based on the fact that $\sum_r |S(x_r)|^2$ is the Hermitian (positive semi-definite) form

$$\sum_m \sum_n a_m \bar{a}_n \sum_r e((m-n)\, x_r),$$

with coefficient matrix PP^*, where P is the $N \times R$ matrix defined by

$$P = [p_{jr}] = [e(jx_r)], \qquad j = 1, \ldots, N; \quad r = 1, 2, \ldots, R.$$

(P^* denotes the complex-conjugate transpose of P.)

Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N$ be the eigenvalues of PP^*. Then it is well known (see Mirsky [**6**; p. 388]) that

$$\sum_r |S(x_r)|^2 \leqslant \lambda_N \sum_n |a_n|^2$$

and that equality occurs when $(a_{M+1}, \ldots, a_{M+N})$ is an eigenvector of PP^* corresponding to λ_N.

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It is not obvious how to derive estimates for λ_N directly from PP^*. However, it is easy to prove that the non-zero eigenvalues of PP^* and P^*P are identical. In Lemma 1 we exhibit a matrix B which is unitarily similar to P^*P. A straight-forward application of Gershgorin's theorem (see Mirsky [6; Theorem 7.5.4, p. 212]) to B then yields Lemma 2, which contains the estimate

$$|\lambda_N - N| < \tfrac{3}{2}\delta^{-1}\log\delta^{-1}.$$

Finally, by a suitable change of variables in the quadratic form corresponding to B, we derive Lemma 3, which contains the estimate

$$|\lambda_N - N| \leqslant \gamma,$$

where γ depends on a certain bilinear form, but does not depend on N. Consequently we have the following

THEOREM. *Let $x_1, x_2, \ldots, x_R$ ($R \geqslant 2$) be any real numbers satisfying*

$$\|x_r - x_s\| \geqslant \delta > 0 \quad \textit{for} \quad r \neq s.$$

Also let $a_{M+1}, \ldots, a_{M+N}$ be arbitrary complex numbers.
Then

$$\text{(a)} \sum_{r=1}^{R} \left| \sum_{n=M+1}^{M+N} a_n e(nx_r) \right|^2 \leqslant (N + \tfrac{3}{2}\delta^{-1}\log\delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2,$$

and

$$\text{(b)} \sum_{r=1}^{R} \left| \sum_{n=M+1}^{M+N} a_n e(nx_r) \right|^2 \leqslant (N + \gamma) \sum_{n=M+1}^{M+N} |a_n|^2,$$

where γ is any number satisfying the inequality

$$\left| \sum_{r} \sum_{\substack{s \\ r \neq s}} \frac{u_r v_s}{\sin \pi(x_s - x_r)} \right| \leqslant \gamma \left(\sum_r u_r^2\right)^{\frac{1}{2}} \left(\sum_s v_s^2\right)^{\frac{1}{2}}$$

for all real numbers $u_1, \ldots, u_R, v_1, \ldots, v_R$.

LEMMA 1. *Let $B = [b_{rs}]$ be the $R \times R$ matrix defined by*

$$b_{rs} = \begin{cases} N & \textit{if} \quad r = s, \\[2ex] \dfrac{\sin N\pi(x_s - x_r)}{\sin \pi(x_s - x_r)} & \textit{if} \quad r \neq s. \end{cases}$$

Also let $P = [p_{jr}]$ be the $N \times R$ matrix defined by

$$p_{jr} = e(jx_r).$$

Then B and P^*P have the same eigenvalues.

Proof.

Let $$A = P^*P = [a_{rs}].$$

Then

$$a_{rs} = \sum_n e(n(x_s - x_r)),$$

and it is easily verified that

$$a_{rs} = \begin{cases} N & \text{if } r = s, \\ e\left(\left(M + \frac{N+1}{2}\right)(x_s - x_r)\right) \dfrac{\sin N\pi(x_s - x_r)}{\sin \pi(x_s - x_r)} & \text{if } r \neq s. \end{cases}$$

Hence if D is the unitary diagonal matrix with diagonal elements

$$e\left(-\left(M + \frac{N+1}{2}\right) x_r\right),$$

it is easily seen that

$$D^*(P^*P)\, D = B.$$

Consequently P^*P and B have the same eigenvalues.

LEMMA 2. *Let $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_R$ be the eigenvalues of the matrix B defined in Lemma* 1. *Then for $r = 1, 2, \ldots, R$ we have*

$$|\mu_r - N| < \tfrac{3}{2}\delta^{-1} \log \delta^{-1}.$$

Proof. By Gershgorin's theorem, applied to B, we have for $r = 1, 2, \ldots, R$

$$|\mu_r - N| \leqslant \max_t \sum_{\substack{s \\ s \neq t}} |b_{ts}|.$$

Now

$$\sum_{\substack{s \\ s \neq t}} |b_{ts}| \leqslant \sum_{\substack{s \\ s \neq t}} \frac{1}{\sin \pi \|x_s - x_t\|}$$

$$\leqslant \sum_{\substack{s \\ s \neq t}} \frac{1}{2\|x_s - x_t\|}.$$

If $R = 2$, the inequality of Lemma 2 easily follows from the last inequality. For $R \geqslant 3$, the last sum has the form

$$\tfrac{1}{2} \sum_{t=1}^{R-1} \|\delta_t\|^{-1},$$

where $\delta \leqslant \delta_1 < \delta_2 < \ldots < \delta_{R-1} \leqslant 1 - \delta$, and $\delta \leqslant \delta_{t+1} - \delta_t$ for $t = 1, \ldots, R-2$. It is then easy to verify that

$$\sum_{t=1}^{R-1} \|\delta_t\|^{-1} \leqslant \|\delta_1\|^{-1} + \|\delta_{R-1}\|^{-1} + \delta^{-1} \int_{\delta_1}^{\delta_{R-1}} \|x\|^{-1}\, dx,$$

by comparison of the areas of suitable rectangles with the area under the curve $y = \|x\|^{-1}$.

Hence

$$\begin{aligned}\sum_{t=1}^{R-1} \|\delta_t\|^{-1} &\leqslant 2\delta^{-1}+\delta^{-1}\int_{\delta}^{1-\delta} \|x\|^{-1}\,dx \\ &= 2\delta^{-1}+2\delta^{-1}\log(\delta^{-1}/2) \\ &\leqslant (2\delta^{-1}\log\delta^{-1})/\log 2 \\ &< 3\delta^{-1}\log\delta^{-1},\end{aligned}$$

completing the proof.

LEMMA 3. *Let γ be any number satisfying the inequality*

$$\left|\sum_{r}\sum_{\substack{s\\ r\neq s}} \frac{u_r v_s}{\sin \pi(x_s-x_r)}\right| \leqslant \gamma\left(\sum_r u_r^2\right)^{\frac12}\left(\sum_s v_s^2\right)^{\frac12} \tag{1}$$

for all real numbers $u_1, \ldots, u_R, \ldots, v_1, \ldots, v_R$. Also let $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_R$ be the eigenvalues of the matrix $B = [b_{rs}]$, where

$$b_{rs} = \begin{cases} N & \text{if } r = s, \\ \dfrac{\sin N\pi(x_s-x_r)}{\sin \pi(x_s-x_r)} & \text{if } r \neq s. \end{cases}$$

Then for $r = 1, 2, \ldots, R$ we have

$$|\mu_r - N| \leqslant \gamma.$$

Proof. Let $z_1, z_2, \ldots, z_R$ be arbitrary real numbers. Also let S be the quadratic form

$$S = \sum_{r}\sum_{\substack{s\\ r\neq s}} z_r z_s \frac{\sin N\pi(x_s-x_r)}{\sin \pi(x_s-x_r)}.$$

Then it is easy to verify that

$$\begin{aligned}S &= 2\sum_{r}\sum_{\substack{s\\ r\neq s}} z_r z_s \frac{\cos N\pi x_r \sin N\pi x_s}{\sin \pi(x_s-x_r)} \\ &= 2\sum_{r}\sum_{\substack{s\\ r\neq s}} \frac{u_r v_s}{\sin \pi(x_s-x_r)},\end{aligned}$$

where $u_r = z_r \cos N\pi x_r$ and $v_s = z_s \sin N\pi x_s$.

Hence by inequality (1)

$$|S| \leqslant 2\gamma \left(\sum_r u_r^2\right)^{\frac{1}{2}} \left(\sum_r v_r^2\right)^{\frac{1}{2}}$$

$$\leqslant \gamma \left(\sum_r u_r^2 + \sum v_r^2\right) = \gamma \sum z_r^2.$$

On taking $(z_1, z_2, \ldots, z_R)$ to be an eigenvector of $B - NI_R$ corresponding to $\mu_r - N$, we deduce that

$$|\mu_r - N| \leqslant \gamma.$$

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