Generalizations of the $3x + 1$ problem and connections with Markov matrices and chains

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3x+1 conjecture (Collatz 1929)

Let $T : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (3x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

Then experimentally, the iterates $x, T(x), T^2(x), \ldots$

(a) with $x > 0$, reach the cycle $1, 2, 1$;

(b) with $x < 0$, reach one of the cycles

$-1, -1$;

$-5, -7, -10, -5$;

$-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34, -17$.

Experiment at
http://www.numbertheory.org/php/collatz.html

A generalization

Let a and b be integers, a even, b odd and

$$T(x) = \begin{cases} (x+a)/2 & \text{if } x \equiv 0 \pmod 2, \\ (3x+b)/2 & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

We expect all iterates to eventually cycle, with finitely many cycles including the following:

(i) a, a;

(ii) $-b, -b$;

(iii) $b+2a, 2b+3a, b+2a$;

(iv) $-5b-4a, -7b-6a, -10b-9a, -5b-4a$;

(v) $-17b-16a, -25b-24a, -37b-36a, -55b-54a,$
$-82b-81a, -41b-40a, -61b-60a, -91b-90a,$
$-136b-135a, -68b-67a, -34b-33a, -17b-16a.$

Experiment at http://www.numbertheory.org/php/generalized_3x+1_cycle.html

The $3x + 371$ mapping

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (3x+371)/2 & \text{if } x \text{ is odd.} \end{cases}$$

We believe there are 9 cycles (lengths in parentheses):

$$0\,(1), -371\,(1), 371\,(2), -1855\,(3), -6307\,(11),$$
$$25\,(222), 265\,(4), 721\,(29), -563\,(14).$$

Experiment at
http://www.numbertheory.org/php/3x+371.html

Hasse's generalization

Let $m > d > 1, \gcd(m, d) = 1$ and let R_d be a set of $d - 1$ nonzero residue classes $\pmod{d}$. Then

$$T(x) = \begin{cases} x/d & \text{if } x \equiv 0 \pmod{d} \\ (mx - r)/d & \text{if } mx \equiv r \pmod{d}, r \in R_d. \end{cases}$$

Then H. Möller conjectured that the sequence of iterates $x, T(x), T^{(2)}(x), \ldots$, eventually cycles for all integers x, if and only if $m < d^{d/(d-1)}$ and that regardless of this inequality, the number of cycles is finite.

Generalized 3x+1 mappings

Let $d \geq 2$ and $m_0, \ldots, m_{d-1}$ be non–zero integers. Also for $i = 0, \ldots, d-1$, let $r_i \in \mathbb{Z}$ satisfy $r_i \equiv im_i \pmod{d}$. Then

$$T(x) = \frac{m_i x - r_i}{d} \qquad \text{if } x \equiv i \pmod{d}$$

defines a mapping $T : \mathbb{Z} \to \mathbb{Z}$, called a *generalized* $3x+1$ *mapping*.

Equivalently, in terms of the integer part symbol,

$$T(x) = \left\lfloor \frac{m_i x}{d} \right\rfloor + a_i \quad \text{if } x \equiv i \pmod{d},$$

where $a_0, \ldots, a_{d-1}$ are integers.

kth iterate formula

If $T^k(x) \equiv i \pmod{d}$, $0 \leq i < d$, we define $m_k(x) = m_i$ and $r_k(x) = r_i$. Then

(a) $$T^k(x) = \frac{m_0(x) \cdots m_{k-1}(x)}{d^k} \left(x - \sum_{i=0}^{k-1} \frac{r_i(x) d^i}{m_0(x) \cdots m_i(x)} \right).$$

(b) If $T^i(x) \neq 0$ for all $i \geq 0$, then

$$T^k(x) = \frac{m_0 \cdots m_{k-1}(x)}{d^k} x \prod_{i=0}^{k-1} \left(1 - \frac{r_i(x)}{m_i(x) T^i(x)} \right).$$

Diophantine equation for a cycle

The kth iterate formula (a) gives the following criterion for $x \in \mathbb{Z}$ to start a cycle of length K with odd iterates $T^{i_t}(x), 0 \le i_1 < \cdots < i_L < K$:

$$(2^K - 3^L)x = \sum_{t=1}^{L} 2^{i_t} 3^{L-t}. \qquad (1)$$

Example. $x = -17$. Here $T^{11}(-17) = -17$ and the iterates $T^k(-17), 0 \le k < 11$ are $-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34$. Hence $i_1 = 0, i_2 = 1, i_3 = 2, i_4 = 3, i_5 = 5, i_6 = 6, i_7 = 7$ and $L = 7$. Then equation (1) gives

$$(2^{11} - 3^7)(-17) = 2363 = 2^0 3^6 + 2^1 3^5 + 2^2 3^4 + 2^3 3^3 + 2^5 3^2 + 2^6 3 + 2^7.$$

Relatively prime maps: Conjectures

Let $\gcd(m_i, d) = 1$ for $0 \leq i \leq d-1$. (The *relatively prime* case).

(i) If $|m_0 \cdots m_{d-1}| < d^d$, then all trajectories $\{T^k(x)\}$, $x \in \mathbb{Z}$, eventually cycle.

(ii) If $|m_0 \cdots m_{d-1}| > d^d$, then almost all trajectories $\{T^k(x)\}$, $x \in \mathbb{Z}$ are divergent (that is, $T^k(x) \to \pm\infty$).

(iii) The number of cycles is finite and positive.

(iv) If the trajectory $\{T^k(x)\}$, $x \in \mathbb{Z}$ diverges, then the iterates are uniformly distributed mod d^α for each $\alpha \geq 1$. i.e.,

$$\lim_{N\to\infty} \tfrac{1}{N}\#\{k < N | T^k(x) \equiv j \pmod{d^\alpha}\} = \frac{1}{d^\alpha}.$$

An example where $|m_0 \cdots m_{d-1}| < d^d$

$$T(x) = \begin{cases} x/3 & \text{if } x \equiv 0 \pmod 3 \\ (2x-2)/3 & \text{if } x \equiv 1 \pmod 3 \\ (13x-2)/3 & \text{if } x \equiv 2 \pmod 3 \end{cases}$$

Here $d = 3, m_0 = 1, m_1 = 2, m_2 = 13$ and $m_0 m_1 m_2 = 26 < 27 = d^d$.

There appear to be six cycles, with starting values $0, 2, 47, -2, -10, -22$.

The trajectory starting with $x = 338$ takes 7161 iterations to reach the cycle beginning with 2. Also the maximum iterate value is $T^{2726}(338)$, a number with 73 digits.

Examples where $|m_0 \cdots m_{d-1}| > d^d$

(1) The $5x+1$ mapping:

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (5x+1)/2 & \text{if } x \text{ is odd,} \end{cases}$$

Here the trajectory starting with $x = 7$ appears to be divergent. There appear to be 5 cycles, with starting values $0, 1, 13, 17, -1$.

(2) (Collatz - a 1–1 map of $\mathbb{Z}$ onto $\mathbb{Z}$):

$$T(x) = \begin{cases} 2x/3 & \text{if } x \equiv 0 \pmod 3 \\ (4x-1)/3 & \text{if } x \equiv 1 \pmod 3 \\ (4x+1)/3 & \text{if } x \equiv 2 \pmod 3 \end{cases}$$

Here the trajectory starting with $x = 8$ appears to be divergent. There appear to be 9 cycles with starting values $0, \pm 1, \pm 2, \pm 4, \pm 44$.

Limiting frequencies conjecture for divergent trajectories (relatively prime T)

For a mapping of relatively prime type, experiments reveal that for each $m > 1$, a divergent trajectory

(a) eventually belongs to a union $B(j_1, m) \cup \ldots \cup B(j_r, m), 0 \leq j_1 < \cdots < j_r \leq m - 1$ of congruence classes (mod m),

(b) occupies each $B(j_i, m)$ with a positive limiting frequency f_i,

(c) occupies each $B(j_i + tm, md), 0 \leq t < d$, with limiting frequency f_i/d.

For a wider class of mappings T, we believe these sets and the frequencies f_i, can be predicted by studying a certain Markov matrix $Q_T(m)$.

An example of limiting frequency behaviour

The $5x - 3$ mapping:

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (5x-3)/2 & \text{if } x \text{ is odd,} \end{cases}$$

(i) $m = 5$. Trajectories such as $\{T^k(-5)\}$ and $\{T^k(-21)\}$ appear to be divergent and eventually occupy the congruence classes $B(1,5), B(2,5), B(3,5), B(4,5)$ with apparent limiting frequencies $8/15, 1/15, 4/15, 2/15$.

(ii) $m = 3$. The trajectory $\{T^k(-5)\}$ occupies $B(1,3)$ and $B(2,3)$ with apparent limiting frequencies $1/2, 1/2$, whereas the trajectory $\{T^k(-21)\}$ occupies $B(1,3)$ for all $k \geq 0$.

Size of divergent trajectory k-th iterate

On the assumption that the limiting frequencies for divergent trajectories exist for the classes $B(j, d)$ and equal $1/d$, the product formula for $T^k(x)$ allows us to us to deduce that

$$|T^k(x)|^{1/k} \to \frac{|m_0 \cdots m_{d-1}|^{1/d}}{d}.$$

If the limiting frequencies f_i exist, but are not uniform, this limit is replaced by

$$|T^k(x)|^{1/k} \to \frac{|m_0|^{f_0} \cdots |m_{d-1}|^{f_{d-1}}}{d}.$$

Some properties of T^{-1}

(i) $T^{-1}(B(j, m))$ is a disjoint union of N congruence classes $\pmod{md}$. Moreover, if $\gcd(m_i, m) = 1$ for $i = 0, \ldots, d-1$, then $N = d$.

(ii) In the relatively prime case, the d^α *cylinders*

$$B(i_0, d) \cap T^{-1}(B(i_1, d)) \cap \cdots \cap T^{-(\alpha-1)}(B(i_{\alpha-1}, d)),$$

$0 \le i_0 < d, \ldots, 0 \le i_{\alpha-1} < d$, are the d^α congruence classes mod d^α.

(iii) In the relatively prime case, if

$$A = B(j, d^\alpha) \text{ and } B = B(k, d^\beta),$$

then $T^{-K}(A) \cap B$ is a disjoint union of $d^{K-\beta}$ congruence classes mod $d^{K+\alpha}$, if $K \ge \beta$.

Extension of T to d-adic integers $\hat{\mathbb{Z}}_d$

We restrict ourselves to the relatively prime case.
T extends uniquely to a continuous mapping $T : \hat{\mathbb{Z}}_d \to \hat{\mathbb{Z}}_d$. This ring is a compact metric space under the d–adic metric and the "congruence" classes mod d^α form a basis for the open sets.
There is a Haar measure μ on the additive group of $\hat{\mathbb{Z}}_d$, where $\mu(B(j, d^\alpha)) = 1/d^\alpha$.

Property (i) implies that $T^{-1}(B(j, d^\alpha))$ is the disjoint union of d congruence classes $\pmod{d^{\alpha+1}}$; hence T is *measure–preserving*:

$$\mu(T^{-1}(A)) = \mu(A),$$

if A is a measurable set in $\hat{\mathbb{Z}}_d$.

Applying the ergodic theorem to $T : \hat{\mathbb{Z}}_d \to \hat{\mathbb{Z}}_d$

Property (iii) of T implies the *strongly–mixing* property

$$\lim_{K\to\infty} \mu(T^{-K}(A) \cap B) = \mu(A)\mu(B)$$

for all measurable sets A and B in $\hat{\mathbb{Z}}_d$; hence T is ergodic:

$$T^{-1}(A) = A \implies \mu(A) = 0 \quad \text{or } 1.$$

Applying the ergodic theorem to $B(j, d^\alpha)$ gives

$$\lim_{N\to\infty} \tfrac{1}{N}\#\{k < N | T^k(x) \equiv j \pmod{d^\alpha}\} = \frac{1}{d^\alpha}$$

for almost all $x \in \hat{\mathbb{Z}}_d$.

H. Möller's d-adic expansion for relatively prime T

For all $x \in \hat{\mathbb{Z}}_d$,

$$x = \sum_{i=0}^{\infty} \frac{r_i(x) d^i}{m_0(x) \cdots m_i(x)}.$$

This tells us that the congruence classes mod d occupied by the iterates of x, in fact determine x.

A corresponding expansion is useful in a later example of a mapping $T : GF(2)[X] \to GF(2)[X]$.

Markov matrix arising from T

To introduce Markov chains, we need a probability space containing $\mathbb{Z}$, which we take to be the *polyadic integers* $\hat{\mathbb{Z}}$. Like the d–adic integers, this ring is a compact metric space that can be defined as a completion of $\mathbb{Z}$. The congruence class $\{x \in \hat{\mathbb{Z}} | x \equiv j \pmod{m}\}$ is also denoted by $B(j, m)$. Then our finitely additive measure μ on $\mathbb{Z}$ extends to a probability Haar measure on $\hat{\mathbb{Z}}$.

Markov chain equation

Then the sequence of random set–valued functions $Y_K(x) = B(T^K(x), m),\ x \in \hat{\mathbb{Z}}$, forms a Markov chain with m states $B(j, m), 0 \leq j < m$ and transition matrix $Q_T(m) = [q_{ij}(m)]$:

$$\begin{aligned} q_{ij}(m) &= Pr\{(T(x) \in B(j, m)|x \in B(i, m)\} \\ &= \mu\{B(i, m) \cap T^{-1}(B(j, m))\}/\mu\{B(i, m) \end{aligned}$$

and Markov property:

$$\begin{aligned} Pr(Y_0(x) = B(i_0, m), \ldots Y_K(x) = B(i_K, m)|\, Y_0(x) = B(i_0, m)) \\ = q_{i_0 i_1}(m) \cdots q_{i_{K-1} i_K}(m). \end{aligned}$$

Markov chain property continued

This last equation is a translation of the statement:

$B(i_0, m) \cap T^{-1}(B(i_1, m)) \cap \cdots \cap T^{-K}(B(i_K, m))$ consists of $p_{i_0 i_1}(m) \cdots p_{i_{K-1} i_K}(m)$ congruence classes $\pmod{md^K}$, where $B(i, m) \cap T^{-1}(B(j, m))$ consists of $p_{ij}(m)$ congruence classes $\pmod{md}$.

The equation also holds if $\gcd(m_i, d^2) = \gcd(m_i, d)$ for $0 \leq i < d$, provided d divides m.

If d divides m, a simple formula exists for $q_{ij}(m)$:

$$q_{ij}(m) = \begin{cases} \frac{\gcd(m_i, d)}{d} & \text{if } T(i) \equiv j \pmod{\frac{m}{d} gcd(m_i, d)}, \\ 0 & \text{otherwise.} \end{cases}$$

A correspondence

With respect to the Markov matrix $Q_T(m)$,

(a) $\mathcal{C}$ is a *closed* set of states if $B \in \mathcal{C}$ and $q_{BB'} > 0$ imply $B' \in \mathcal{C}$.

(b) $\mathcal{C}$ is a *positive recurrent* set of states if it is a minimal closed set.

Then under the corrrespondence

$$S_{\mathcal{C}} = B(j_1, m) \cup \cdots \cup B(j_t, m) \leftrightarrow \mathcal{C} = \{B(j_1, m), \ldots, B(j_t, m)\},$$

where $0 \leq j_1 < \cdots < j_t < m$,

(a) T–invariant sets $S_{\mathcal{C}}$ correspond to closed sets $\mathcal{C}$,

(b) minimal T–invariant sets $S_{\mathcal{C}}$ (*ergodic sets*) correspond to positive recurrent classes $\mathcal{C}$,

Structure of the ergodic sets $S_{\mathcal{C}}$

Let $\mathcal{N}_1$ be the set of positive integers composed of primes which divide at least one m_i; also let $\mathcal{N}_2$ be the set of positive integers which are relatively prime to each m_i.
Also, for $0 \leq i < j < d$ let

$$\Delta_{ij} = r_j(d - m_i) - r_i(d - m_j)$$

and $\Delta = \gcd_{0 \leq i < j < d} \Delta_{ij}$.

Let $S_1^{(m)}, \ldots, S_{r(m)}^{(m)}$ be the ergodic sets $\pmod{m}$. Then the following are all the ergodic sets:

(a) $\hat{\mathbb{Z}}$ if $m \in \mathcal{N}_2$ and $\gcd(m, \Delta) = 1$;

(b) $S_1^{(m)}, \ldots, S_{r(m)}^{(m)}$, where $m | \Delta$, $m \in \mathcal{N}_2$;

(c) $S_1^{(m)}$, where $m \in \mathcal{N}_1$;

(d) any intersection of a set of type (b) and one of type (c).

A mapping property of ergodic sets

Suppose T is a mapping of relatively prime type.

If m divides n and $B(j_1, n) \cup \cdots \cup B(j_t, n)$ is an ergodic set $\pmod{n}$, then $B(j_1, m) \cup \cdots \cup B(j_t, m)$ is an ergodic set $\pmod{m}$.

A formula for the stationary distribution $\rho_B, B \in \mathcal{C}$

Let $p_{Kij}(m)$ be the number of congruence classes $\pmod{md^K}$ contained in $B(i,m) \cap T^{-K}(B(j,m))$.
Then the cylinder equation implies

$$[p_{Kij}] = [p_{ij}]^K = d^K \{Q_T(m)\}^K.$$

Hence

$$\frac{\mu\{B(i,m) \cap T^{-K}(B(j,m))\}}{\mu\{B(i,m)\}} = p_{Kij}(m)/d^K = [\{Q_T(m)\}^K]_{ij}.$$

Then if $B(j,m)$ belongs to $\mathcal{C}$, by the well–known limit result for Markov matrices, summing over $B(i,m) \in \mathcal{C}$, we get

$$\rho_{B(j,m)} = lim_{N\to\infty} \frac{1}{N} \sum_{K<N} \frac{\mu\{S_{\mathcal{C}} \cap T^{-K}(B(j,m))\}}{\mu\{S_{\mathcal{C}}\}}.$$

Ergodic property

Let $\mathcal{C}$ be a positive recurrent class and for each $B \in \mathcal{C}$, let ρ_B be the component of the unique stationary distribution over $\mathcal{C}$. Then $S_{\mathcal{C}} = \cup_{B \in \mathcal{C}}$ is T–invariant. Hence an ergodic theorem for Markov chains, applied to the $Y_n(x)$ restricted to $S_{\mathcal{C}}$, gives for a $B \in \mathcal{C}$:

$$Pr\left(\lim_{K\to\infty} \tfrac{1}{K}\#\{n; n < K,\ Y_n(x) = B\} = \rho_B\right) = 1.$$

In other words,

$$\lim_{N\to\infty} \tfrac{1}{N}\#\{k; k < N,\ T^k(x) \in B\} = \rho_B$$

for almost all $x \in S_{\mathcal{C}}$.

Transient class property

Let $\mathcal{P}$ be the set of positive recurrent states. Then

$$\Pr(Y_n(x) \in \mathcal{P} \text{ for some } n > 0) = 1.$$

Hence we expect all divergent trajectories starting in a *transient* $B(j, m)$ to eventually enter some *ergodic* set $S_{\mathcal{C}}$, occupying each $B \in S_{\mathcal{C}}$ with limiting frequency $\rho(B)$.

Ergodic sets (mod d)

In the case of relatively prime T, there is only one positive recurrent class, $\mathcal{C}_1 = \{B(0,d),\ldots,B(d-1,d)\}$. However for non relatively prime T, where $\gcd(m_i,d^2) = \gcd(m_i,d)$, $0 \le i < d$,, we may have several such classes $\mathcal{C}_1,\ldots,\mathcal{C}_r$ (and some transient states). We expect

(i) if $\prod_{B_j \in \mathcal{C}_i} \left(\frac{|m_j|}{d}\right)^{\rho_{B_j}} < 1$, then all trajectories starting in $S_{\mathcal{C}_i}$ will enter a cycle.

(ii) if $\prod_{B_j \in \mathcal{C}_i} \left(\frac{|m_j|}{d}\right)^{\rho_{B_j}} > 1$, then almost all trajectories starting in $S_{\mathcal{C}_i}$ will be divergent.

Example 1 of $Q_T(m)$

The $5x-3$ mapping. Here

$$Q_T(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

There are two positive recurrent classes:

$$\mathcal{C}_1 = \{B(0,3)\} \text{ and } \mathcal{C}_2 = \{B(2,3), B(2,3)\}.$$

The stationary distribution for $\mathcal{C}_2$ is $1/2, 1/2$. Trajectory $\{T^k(-5)\}$ appears to diverge and occupies $B(1,3)$ and $B(2,3)$ with limiting frequencies $1/2, 1/2$. Trajectory $\{T^k(-21)\}$ appears to diverge and occupies $B(0,3)$ for all $k \geq 0$.

Example 2 of $Q_T(m)$

A four–branched mapping:

$$T(x) = \begin{cases} 3x/2 & \text{if } x \equiv 0 \pmod 4 \\ (x+1)/2 & \text{if } x \equiv 1 \pmod 4 \\ x/2+1 & \text{if } x \equiv 2 \pmod 4 \\ (5x+3)/2 & \text{if } x \equiv 3 \pmod 4 \end{cases}$$

$$Q_T(4) = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

on interchanging rows and columns 2 and 3.

Example 2 continued

There are two positive recurrent classes:

$$\mathcal{C}_1 = \{B(0,4), B(2,4)\} \text{ and } \mathcal{C}_2 = \{B(1,4), B(3,4)\}.$$

The stationary vectors for both classes are $(1/2, 1/2)$. Then

$$\prod_{B_j \in \mathcal{C}_1} \left(\frac{|m_j|}{d} \right)^{\rho_{B_j}} = (3/2)^{1/2}(1/2)^{1/2} < 1,$$

$$\prod_{B_j \in \mathcal{C}_2} \left(\frac{|m_j|}{d} \right)^{\rho_{B_j}} = (1/2)^{1/2}(5/2)^{1/2} > 1.$$

Hence we expect all trajectories starting with an even integer to enter one of the cycles with starting values $0, 2, 4, -8, -32$, while most starting with an odd trajectory should diverge or else enter one of the cycles with starting values
$-1, 1, 3, -5, 7, 79, 87, 103, 107, 123.$

Example 3 of $Q_T(m)$

$$T(x) = \begin{cases} x/3 - 1 & \text{if } x \equiv 0 \pmod 3 \\ (x+5)/3 & \text{if } x \equiv 1 \pmod 3 \\ 10x - 5 & \text{if } x \equiv 2 \pmod 3 \end{cases}$$

$$Q_T(3) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}.$$

$(Q_T(3))^2$ is positive, so all states are positive recurrent, with stationary distribution $1/2, 1/4, 1/4$. Also

$$(1/3)^{1/2}(1/3)^{1/4}(30/3)^{1/4} < 1.$$

Hence we expect all trajectories to eventually cycle. In fact there appear to be five cycles, starting with values There appear to be five cycles, with starting values 0, 5, 17, -1, -4.

Example 4 of $Q_T(m)$

$$T(x) = \begin{cases} x & \text{if } x \equiv 0 \pmod 3 \\ (7x+2)/3 & \text{if } x \equiv 1 \pmod 3 \\ (x-2)/3 & \text{if } x \equiv 2 \pmod 3 \end{cases}$$

$$Q_T(3) = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

There is one positive recurrent class $\mathcal{C}_1 = \{B(0,3)\}$ and transient states $B(1,3)$ and $B(2,3)$.

Here $3|x$ implies $3|T(x)$; so once a trajectory enters the zero residue class mod 3, it remains there. Experimental evidence (http://www.numbertheory.org/php/markov.html) strikingly suggests that if $T^k(x) \equiv \pm 1 \pmod 3$ for all $k \geq 0$, then the trajectory must eventually enter one of the cycles $-1, -1$ or $-2, -4, -2$. The author offers a \$100 (Australian) prize for a proof. This problem seems just as intractable as the $3x+1$ problem, but is more spectacular.

The general mapping T

In 1983, George Leigh introduced a Markov chain $\{Y_n\}$, which enabled predictions to be made (mod m), $d \mid m$, for a wider class of T.

Let $m_i = b_i d_i$, where $b_i \in \mathbb{Z},\ d_i \in \mathbb{N}$ and $\gcd(d, b_i) = 1$, where d_i divides some power of d, $0 \le i < d$.

We define a sequence of random functions on $\hat{\mathbb{Z}} : x \to Y_n(x) \in \mathcal{B}$, the collection of congruence classes of the form $B(j, mk)$, where k divides some power of d:

The random set–valued functions

(a) $Y_0(x) = B(x, m)$;

(b) $Y_{n+1}(x) = B(T^{n+1}(x), mk_{n+1})$, where

$$k_0 = 1, \quad k_{n+1} = \frac{d_j k_n}{\gcd(d_j k_n, d)}$$

and $T^n(x) \equiv j \pmod{d}$, $0 \le j < d$.

Note that if $\gcd(m_j, d) = 1$ for $0 < j < d$ or $\gcd(m_j, d^2) = \gcd(m_j, d)$ (i.e., $d_j \mid d$) for $0 < j < d$, then $k_n = 1$ for all n.

Markov property and Transition probabilities $q_{BB'}$

We have

$$Pr(Y_0(x) = B_0, \dots Y_K(x) = B_K | Y_0(x) = B_0) \\ = q_{B_0 B_1} \cdots q_{B_{K-1} B_K},$$

where the transition probabilities $q_{BB'}$ are defined as follows:

Let $B = B(j, M)$, $B' = B(j', M')$, $N = Md_j/d, N' = \mathrm{lcm}(N, m)$. Then

$$q_{BB'} = Pr(Y_{n+1}(x) = B' | Y_n(x) = B) \\ = \begin{cases} \frac{\gcd(Md_j/m, d)}{d} & \text{if } B' = B(T(j) + tN'/N, N'), \\ 0 & \text{otherwise.} \end{cases}$$

Algorithm for computing the states reached and the $q_{BB'}$

Starting with initial state $B = B(j, M)$ equal to one of $B(0, m), \ldots, B(m-1, m)$, form the N'/N states

$$B' = B(T(j) + tN'/N, N'), 0 \leq t < N'/N.$$

These give the states B' with $q_{BB'} > 0$; also $q_{BB'} = N/N'$.

If the process finishes and n states are produced, we get an $n \times n$ transition matrix $Q_T(m)$, for which the row corresponding to state B has N'/N non–zero entries, each equal to N/N'.

Criteria for cycling and divergence

Suppose that the Markov chain for $m = d$ has finitely many states. Also if $\mathcal{C}$ be a positive recurrent class, for each $B \in \mathcal{C}$, let ρ_B be the corresponding limiting probability. Then

(a) Every divergent trajectory will eventually occupy each class B of some positive class $\mathcal{C}$, with limiting frequency ρ_B.

(b) Let $\mathcal{C}$ be a positive recurrent class for the Markov chain $\pmod d$ and let

$$p_j = \sum_{\substack{B \in \mathcal{C} \\ B \subseteq B(j, d)}} \rho_B .$$

Criteria for cycling and divergence continued

Then if

$$\prod_{B(j,d)\in\mathcal{C}} \left(\frac{|m_j|}{d}\right)^{p_j} < 1,$$

all trajectories starting in a $B(j,d) \in \mathcal{C}$ will eventually cycle.
However if

$$\prod_{B(j,d)\in\mathcal{C}} \left(\frac{|m_j|}{d}\right)^{p_j} > 1,$$

almost all trajectories starting in a $B(j,d) \in \mathcal{C}$ will diverge.

Example 1 (Leigh 1983)

Let $T : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ 12x + 4 & \text{if } x \text{ is odd.} \end{cases}$$

Here $d = 2, m_0 = 1, m_1 = 24$. Then $d_0 = 1, d_1 = 8$ and $\gcd(m_1, d_1^2) = \gcd(24, 4) = 4 \neq \gcd(m_1, d) = 2$.
The recursive scheme for generating the states and positive transition probabilities:

$$\begin{array}{rcl} B(0,2) & \to & B(0,2) \\ & \to & B(1,2) \\ B(1,2) & \to & B(0,8) \\ B(0,8) & \to & B(0,4) \\ B(0,4) & \to & B(0,2). \end{array}$$

States: $B(0,2), B(1,2), B(0,8), B(0,4)$.

Example 1 continued

$$Q_T(2) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$Q_T(2)^6 > 0$, so the states $B(0,2), B(1,2), B(0,8), B(0,4)$ form a positive recurrent class with stationary vector

$$(\rho_{B(0,2)}, \rho_{B(1,2)}, \rho_{B(0,8)}, \rho_{B(0,4)}) = (2/5, 1/5, 1/5, 1/5).$$

Then with $p_0 = \rho_{B(0,2)} + \rho_{B(0,8)} + \rho_{B(0,4)}$ and $p_1 = \rho_{B(1,2)}$,

$$\begin{aligned}(m_0/d)^{p_0}(m_1/d)^{p_1} &= (1/2)^{2/5+1/5+1/5}(24/2)^{1/5} \\ &= (3/4)^{1/5} < 1,\end{aligned}$$

so we expect all trajectories to enter cycles.

An example of Leigh (1986)

$$T(x) = \begin{cases} x/4 & \text{if } x \equiv 0 \pmod 8 \\ (x+1)/2 & \text{if } x \equiv 1 \pmod 8 \\ 20x - 40 & \text{if } x \equiv 2 \pmod 8 \\ (x-3)/8 & \text{if } x \equiv 3 \pmod 8 \\ 20x + 48 & \text{if } x \equiv 4 \pmod 8 \\ (3x-13)/2 & \text{if } x \equiv 5 \pmod 8 \\ (11x-2)/4 & \text{if } x \equiv 6 \pmod 8 \\ (x+1)/8 & \text{if } x \equiv 7 \pmod 8 \end{cases}$$

We find there are 9 states in the Markov chain mod 8:

$B(0,8)$, $B(1,8)$, $B(2,8)$, $B(3,8)$, $B(4,8)$, $B(5,8)$, $B(6,8)$, $B(7,8)$, $B(0,32)$,

$$\begin{array}{rcl} B(0,8) & \to & B(0;2;4;6,8) \\ B(1,8) & \to & B(1;5,8) \\ B(2,8) & \to & B(0,32) \\ B(3,8) & \to & B(0;1;2;3;4;5;6;7,8) \\ B(4,8) & \to & B(0,32) \\ B(5,8) & \to & B(1;5,8) \\ B(6,8) & \to & B(0;2;4;6,8) \\ B(7,8) & \to & B(0;1;2;3;4;5;6;7,8) \\ B(0,32) & \to & B(0,8) \end{array}$$

An example of Leigh (1986) continued

There are two positive recurrent classes: $\mathcal{C}_1 = \{B(1,8),\ B(5,8)\}$ and $\mathcal{C}_2 = \{B(0,8),\ B(0,32),\ B(2,8),\ B(4,8),\ B(6,8)\}$, with transient states $B(3,8)$ and $B(7,8)$.

The limiting probabilities are $\rho_1 = (\frac{1}{2}, \frac{1}{2})$ and $\rho_2 = (\frac{3}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$, respectively.

We have $p_1 = p_5 = \frac{1}{2}$ and as

$$\prod_{B_j \in \mathcal{C}_1} \left(\frac{|m_j|}{d} \right)^{p_j} = (1/2)^{1/2} (3/2)^{1/2} < 1,$$

we expect every trajectory starting in $\mathcal{S}_{\mathcal{C}_1} = B(1,8) \cup B(5,8)$ to cycle, reaching one of $1, 13, 61, 205, -11$.

An example of Leigh (1986) finished

Also $p_0 = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$ and $p_2 = p_4 = p_6 = \frac{1}{8}$. Then as

$$\prod_{B_j \in \mathcal{C}_2} \left(\frac{|m_j|}{d} \right)^{p_j} = (1/4)^{5/8} 20^{1/8} 20^{1/8} (11/4)^{1/8} > 1,$$

we expect most trajectories starting in $\mathcal{S}_{\mathcal{C}_2} = B(0,2)$ to diverge, displaying frequencies $\rho_2 = (\frac{5}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ in the respective component congruence classes. For example, the trajectory starting with 46.

We found 8 cycles lying in $B(0,2)$, with starting values 0, 10, 158, 3292, 4244, −2, −12, −18.

An example of Venturini (1992)

$$T(x) = \begin{cases} 2500x/6 + 1 & \text{if } x \equiv 0 \pmod 6 \\ (21x - 9)/6 & \text{if } x \equiv 1 \pmod 6 \\ (x + 16)/6 & \text{if } x \equiv 2 \pmod 6 \\ (21x - 51)/6 & \text{if } x \equiv 3 \pmod 6 \\ (21x - 72)/6 & \text{if } x \equiv 4 \pmod 6 \\ (x + 13)/6 & \text{if } x \equiv 5 \pmod 6). \end{cases}$$

There are 9 states in the Markov chain (mod 6):

$$\begin{array}{lcl} B(0,6) & \to & B(1,12), B(5,12), B(9,12) \\ B(1,6) & \to & B(2,6), B(5,6) \\ B(2,6) & \to & B(0,6), B(1,6), B(2,6), B(3,6), B(4,6), B(5,6) \\ B(3,6) & \to & B(2,6), B(5,6) \\ B(4,6) & \to & B(2,6), B(5,6) \\ B(5,6) & \to & B(0,6), B(1,6), B(2,6), B(3,6), B(4,6), B(5,6) \\ B(1,12) & \to & B(2,6) \\ B(5,12) & \to & B(1,6), B(3,6), B(5,6) \\ B(9,12) & \to & B(5,6), \end{array}$$

namely $B(0,6), B(1,6), B(2,6), B(3,6), B(4,6), B(5,6), B(1,12), B(5,12), B(9,12)$.

Venturini example finished

The 9 states form a positive recurrent class with limiting probabilities

$$\rho = \left(\tfrac{18}{202}, \tfrac{20}{202}, \tfrac{53}{202}, \tfrac{20}{202}, \tfrac{18}{202}, \tfrac{55}{202}, \tfrac{6}{202}, \tfrac{6}{202}, \tfrac{6}{202}\right).$$

Noting that $B(1,12) \subseteq B(1,6)$, $B(9,12) \subseteq B(3,6)$, $B(5,12) \subseteq B(5,6)$, we get

$$p_0 = \rho_{B(0,6)},\ p_1 = \rho_{B(1,12)} + \rho_{B(1,6)},\ p_2 = \rho_{B(2,6)},$$
$$p_3 = \rho_{B(9,12)} + \rho_{B(3,6)},\ p_4 = \rho_{B(4,6)},\ p_5 = \rho_{B(5,12)} + \rho_{B(5,6)}.$$

Then $\prod_{i=0}^{d-1} (m_i/d)^{p_i} < 1$ and we expect all trajectories to eventually cycle. There appear to be two cycles, with starting values 2 and 6.

http://www.numbertheory.org/php/venturini1.html

Example of infinitely many states (Chris Smyth 1993)

$$T(x) = \begin{cases} 3x/2 & \text{if } x \equiv 0 \pmod 2 \\ \lfloor 2x/3 \rfloor & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

This can be regarded as a 6–branched mapping. The integer trajectories are much simpler to describe than the Markov chain:

(i) A non–zero even integer $2^r(2c+1)$ is successively multiplied by $3/2$ until it reaches $3^{r+2}(2c+1) = 6k+3$.

(ii) $6k+3 \to 4k+2 \to 6k+3$.

(iii) $6k+1 \to 4k \to 6k \to 9k \to 6k$.

(iv) $6k+5 \to 4k+3$ and unless we encounter 0 or -1 (fixed points), we must eventually reach $B(1,6)$ or $B(3,6)$.

With $m = 6$, there are infinitely many states. e.g., $Y_n(0) = B(0, 2 \cdot 3^{n+1})$ for $n \geq 0$.

Other rings: $GF(2)[x]$

Here the conjectural picture for trajectories is not so clear. Here is an example of relatively prime type where $|m_0 \cdots m_{|d|-1}| = |d|^{|d|}$, where $|f| = 2^{\deg f}$.

$$T(f) = \begin{cases} \frac{f}{x} & \text{if } f \equiv 0 \pmod{x} \\ \frac{(x^2+1)f+1}{x} & \text{if } f \equiv 1 \pmod{x} \end{cases}$$

Most trajectories appear to cycle. However the trajectory starting from $1 + x + x^3$ exhibits a regularity which enabled its divergence to be proved: If $L_n = 5(2^n - 1)$, then

$$T^{L_n}(1 + x + x^3) = \frac{1 + x^{3 \cdot 2^n + 1} + x^{3 \cdot 2^n + 2}}{1 + x + x^2}.$$

The figure next page, shows the first 38 iterates.

Divergent trajectory $\{T^k(1+x+x^3)\}$ in $GF_2[x]$

The first 38 iterates

```
 0:1101
 1:11001
 2:111101
 3:1001001
 4:01101101
 5:1101101 ←
 6:11011001
 7:110111101
 8:1101001001
 9:11001101101
10:111111011001
11:1000010111101
12:01001001001001
13:1001001001001
14:01101101101101
15:1101101101101 ←
16:11011011011001
17:110110110111101
18:1101101101001001
19:11011011001101101
20:110110111111011001
21:1101101000010111101
22:11011001001001001001
23:110111101101101101101
24:1101001011011011011001
25:11001100110110110111101
26:111111111101101101001001
27:1000000001011011001101101
28:01000000100110111111011001
29:1000000100110111111011001
30:01000010111101000010111101
31:1000010111101000010111101
32:01001001001001001001001001
33:1001001001001001001001001
34:01101101101101101101101101
35:1101101101101101101101101 ←
36:11011011011011011011011001
37:110110110110110110110111101
```

Polynomials over GF(2) continued

There are infinitely many cycles, many of which have no recognisable pattern.

However the trajectories starting with

$$g_n = (1 + x^{2^n - 1})/(1 + x) = 1 + x + \cdots + x^{2^n - 2}$$

possess symmetry and are purely periodic, with period–length 2^n.

Cyclic trajectory: $g_4(x) = (1 + x^{15})/(1 + x) \in GF(2)[x]$

```
 0:111111111111111
 1:1000000000000011
 2:01000000000001111
 3:1000000000001111
 4:01000000000110011
 5:1000000000110011
 6:01000000011111111
 7:1000000011111111
 8:01000001100000011
 9:1000001100000011
10:01000111000001111
11:1000111000001111
12:01011001100110011
13:1011001100110011
14:00111111111111111
15:0111111111111111
16:111111111111111
```

Mappings of rings of algebraic integers

Let d be a non–unit in the ring O_K of integers of an algebraic number field K. Then O_K is composed of $|Norm_k(d)|$ congruence classes (mod d) and we can consider generalized $3x+1$ mappings $T : O_K \to O_K$. The conjectural picture for trajectories is not entirely clear.

Example 1 (Leigh 1983). $T : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$ is defined by

$$T(\alpha) = \begin{cases} \alpha/\sqrt{2} & \text{if } \alpha \equiv 0 \pmod{\sqrt{2}} \\ (3\alpha+1)/\sqrt{2} & \text{if } \alpha \equiv 1 \pmod{\sqrt{2}}. \end{cases}$$

Equivalently, write $\alpha = x + y\sqrt{2}$, where $x, y \in \mathbb{Z}$. Then

$$T(x, y) = \begin{cases} (y, x/2) & \text{if } x \equiv 0 \pmod 2 \\ (3y, (3x+1)/2) & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

There appear to be finitely many cycles with starting values

$$0, 1, -1, -5, -17, -2-3\sqrt{2}, \quad -3-2\sqrt{2}, \quad 9+10\sqrt{2}.$$

Example 1 continued

An interesting feature is the presence of at least three one–dimensional T–invariant sets S_1, S_2, S_3 in $\mathbb{Z} \times \mathbb{Z}$:

(i) $S_1: \; x = 0$ or $y = 0$,

(ii) $S_2: \; 2x + y + 1 = 0$ or $x + 4y + 1 = 0$,

(iii) $S_3: \; x + y + 1 = 0$ or $x + 2y + 1 = 0$ or $x + 2y + 2 = 0$.

Trajectories starting in S_1 or S_2 oscillate from one line to the other, while those starting in S_3 oscillate between the first and either of the second and third.

Trajectories starting in S_1 will cycle, as $T^2(x, 0) = (C(x), 0)$ and $T^2(0, y) = (0, C(y))$, where C denotes the $3x + 1$ mapping.

Example 2

$T : \mathbb{Z}[\sqrt{3}] \to \mathbb{Z}[\sqrt{3}]$ is defined by

$$T(x) = \begin{cases} x/\sqrt{3} & \text{if } x \equiv 0 \pmod{\sqrt{3}} \\ (x-1)/\sqrt{3} & \text{if } x \equiv 1 \pmod{\sqrt{3}} \\ (4x+1)/\sqrt{3} & \text{if } x \equiv 2 \pmod{\sqrt{3}} \end{cases}$$

There are at least 103 cycles. The trajectory starting with $-1 - 5\sqrt{3}$ appears to be divergent. Divergent trajectories produce limiting frequencies approximating $(\cdot 27, \cdot 32, \cdot 40)$ in the residue classes $0, 1, 2 \pmod{\sqrt{3}}$. Interpretation?

Website

- http://www.numbertheory.org/php/collatz.html