X-RAYS OF THE RIEMANN ZETA AND XI FUNCTIONS

by

James M. Hill and Robert K. Wilson,

School of Mathematics and Applied Statistics,

University of Wollongong,

Wollongong, NSW 2522 Australia

email: jhill@uow.edu.au

Contents

X-RAYS OF THE RIEMANN ZETA

AND XI FUNCTIONS

J. M. HILL [1] AND R. K. WILSON

Abstract. The X-ray of a function of a complex variable are the two families of curves along which the function is either purely real or purely imaginary. Following previous work, a thicker black line is used to show those curves along which the imaginary part of the function vanishes (i.e. the function is real along such curves), while a thinner grey line designates those curves for which the real part of the function vanishes (i.e. the function is purely imaginary along such curves). The X-ray of the Riemann zeta function $\zeta(s)$ has been given and fully delineated by a previous author. Here we show that a considerably simplified picture emerges for the X-rays of the Riemann xi function $\xi(s)$ and related functions. The pictures so obtained are entirely consistent with the validity of the Riemann hypothesis, while the X-ray of the function $\log \xi(s)$ produces an unusual situation.

1. Introduction

Perhaps the most famous outstanding problem of modern mathematics is the Riemann hypothesis, formulated by Bernhard Riemann in 1859 (see Edwards [4]). For $s = x + iy$ and $x > 1$ the zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1}$$

and the following equation due to Euler

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.2}$$

where p ranges over all primes ($p = 2, 3, 5, 7, 11, \ldots$), indicates the existence of a deep-lying connection between the theory of the zeta function and prime number theory. The Riemann hypothesis asserts that, "all the complex zeros of the function which is the analytical continuation of (1.1) in the region $0 < \Re(s) < 1$ (now known as the critical strip) lie on the line $x = 1/2$".

In the 1974 book by Edwards [4] (page 164) states: "Even today, more than a hundred years later, one cannot really give any solid reasons for saying that the truth of the Riemann hypothesis is probable. The theorem of Bohr and Landau stating that for any $\delta > 0$ all but an infinitesimal proportion of the roots ρ lie within δ of $\Re(s) = 1/2$ is the only positive result which lends real evidence to the hypothesis. Also the verification of the hypothesis for the first three and a half million roots above the real axis perhaps makes it more probable. However, any real reason, any plausibility argument or heuristic basis for the statement, seems entirely lacking". Edwards repeats this sentiment (page 268): "One of the things which makes the Riemann hypothesis so difficult is the fact that there is no plausibility argument, no hint of a reason, however unrigorous, why it should be true". Actually, quite recently the Riemann hypothesis has been numerically verified for the first fifty billion zeros (contact Sebastian Wedeniwski, email wedeniws@de.ibm.com). A highly readable and comprehensive account of such recent developments is given by Conrey [3].

---

[1]School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, N.S.W. 2522, Australia

In terms of understanding the subtleties of the Riemann zeta function and its analytical continuation, Arias-de-Reyna has proposed the notion of the X-ray of the function [1]. The X-ray of a function of a complex variable are the two families of curves along which the function is either purely real or purely imaginary. A thicker black line is used to show those curves along which the imaginary part of the function vanishes (i.e. the function is real along such curves), while a thinner grey line designates those curves for which the real part of the function vanishes (i.e. the function is purely imaginary along such curves). Arias-de-Reyna shows that these pictorial devices provide an excellent means of showing in a single picture, information which is otherwise scattered. The purpose of this note is to extend this approach, and in particular show that the X-rays of the Riemann xi function present an increasingly simplified presentation of this information. In addition, we show that the X-ray for the function $\log \xi(s)$ gives an unexpected outcome. The computer software program Mathematica has been used to draw these implicit function curves. This is based on calculating a set of function values, and drawing a curve along the boundary between the positive and negative values.

In the following section we present a brief derivation of the basic equations. In the section thereafter we state the basic symmetries of the Riemann xi functions, which together with the Cauchy-Riemann equations for an analytic function of a complex variable, constitute the elementary theoretical tools necessary to make credible the X-ray diagrams obtained numerically. In Section 4 we present X-rays for the four functions $\zeta(s)$, $\xi^*(s)$, $\xi(s)$ and $\log \xi(s)$, which are all defined in the following section.

2. Basic equations

We may show that the function $\zeta(s)$ which is the appropriate analytical continuation of $\zeta(s)$, as defined by either (1.1) or (1.2), may be determined from the Mellin transform expression

$$\xi^*(s) = \frac{\Gamma(s/2)\zeta(s)}{\pi^{s/2}} = \int_0^\infty t^{s/2-1} \phi(t) dt = \xi^*(1-s) \tag{2.1}$$

which is valid for all s, where $\phi(t)$ is the function defined by

$$\phi(t) = \sum_{n=1}^\infty e^{-\pi n^2 t} - \frac{1}{2\sqrt{t}}, \tag{2.2}$$

and $\Gamma(s)$ denotes the usual gamma function, which for $\Re(s) > 0$ is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt. \tag{2.3}$$

For $0 \leq t < \infty$, the function $\phi(t)$ is a well behaved monotonically increasing function such that $-1/2 \leq \phi(t) < 0$ (see Figure 1), which is known to satisfy the functional equation

$$\phi(1/t) = \sqrt{t}\phi(t), \tag{2.4}$$

which may be expressed in the invariant form

$$(1/t)^{1/4}\phi(1/t) = t^{1/4}\phi(t). \tag{2.5}$$

We comment that the final equality of equation (2.1) follows immediately from the substitution $t \to 1/t$ and the functional equation (2.4). The above equations can be derived from the well known representation of the zeta function (see Edwards [4] page 16 or Whittaker and Watson [8] page 273)

$$\xi^*(s) = \int_1^\infty \psi(t) \left\{ t^{s/2} + t^{(1-s)/2} \right\} \frac{dt}{t} - \frac{1}{s(1-s)}, \tag{2.6}$$

4

which is valid for all s, and $\psi(t)$ denotes simply

$$\psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}, \tag{2.7}$$

which is known to satisfy the functional equation

$$\psi(1/t) + 1/2 = \sqrt{t}\left[\psi(t) + 1/2\right]. \tag{2.8}$$

As described in Hill [5], and using the functional equation (2.8), equation (2.6) can be rearranged to yield (2.1), where the final equality follows from the substitution $t \to 1/t$ and using the functional equation (2.5). Alternatively, equation (2.1) follows from an immediate application of the Muntz formula (see for example, Titchmarsh [7], page 28). On making the substitution $t \to 1/t$, it is clear from (2.1) and (2.5) that we have

$$\xi^*(s) = \int_0^{\infty} t^{(s-1/2)/2} t^{1/4} \phi(t) \frac{dt}{t} = \int_0^{\infty} t^{-(s-1/2)/2} t^{1/4} \phi(t) \frac{dt}{t},$$

and therefore by addition we obtain

$$\xi^*(s) = \int_0^{\infty} \cosh\left[\frac{1}{2}\left(s - \frac{1}{2}\right)\log t\right] t^{1/4} \phi(t) \frac{dt}{t}, \tag{2.9}$$

which shows that $\xi^*(s)$ is an even function of $s - 1/2$. Further, with $s = x + iy$ we may deduce

$$\begin{aligned}
\xi^*(s) &= \int_0^{\infty} \cosh\left[\frac{1}{2}\left(x - \frac{1}{2}\right)\log t\right] \cos\left(\frac{y}{2}\log t\right) t^{1/4} \phi(t) \frac{dt}{t} \\
&+ i \int_0^{\infty} \sinh\left[\frac{1}{2}\left(x - \frac{1}{2}\right)\log t\right] \sin\left(\frac{y}{2}\log t\right) t^{1/4} \phi(t) \frac{dt}{t}, \tag{2.10}
\end{aligned}$$

which shows in particular that $\Im(\xi^*(s))$ vanishes along $x = 1/2$.

3. Symmetries of the Riemann xi functions

Rather than focussing on $\xi^*(s)$ defined by (2.1), it is clear from the X-rays given in the following section that the xi function used by Riemann himself (see for example Edwards [4], page 301), namely

$$\xi(s) = \frac{1}{2}\, s\, (s - 1)\, \xi^*(s) = \frac{1}{2}\, s\, (s - 1)\, \frac{\Gamma(s/2)\, \zeta(s)}{\pi^{s/2}}, \tag{3.1}$$

turns out to exhibit the simpler X-ray. However, both xi functions $\xi^*(s)$ and $\xi(s)$ have the same symmetries discussed below, but here for definiteness we will discuss the Riemann xi function $\xi(s)$, which we assume has real and imaginary parts $u(x, y)$ and $v(x, y)$, thus

$$\xi(s) = u(x, y) + iv(x, y), \tag{3.2}$$

where $s = x + iy$. The specific analytical forms of $u(x, y)$ and $v(x, y)$ may be determined from (2.10) and (3.1), thus

$$\begin{aligned}
u(x, y) &= \frac{1}{2}\left[x(x - 1) - y^2\right] u^*(x, y) - y\left(x - \frac{1}{2}\right) v^*(x, y), \\
v(x, y) &= \frac{1}{2}\left[x(x - 1) - y^2\right] v^*(x, y) + y\left(x - \frac{1}{2}\right) u^*(x, y), \tag{3.3}
\end{aligned}$$

5

where $u^*(x, y)$ and $v^*(x, y)$ denote the integral expressions given in (2.10). From (2.10) and (3.3) it is clear that again $\Im(\xi(s))$ vanishes along $x = 1/2$. Further, from (2.1) and (3.1) it is clear that $\xi(s)$ satisfies the functional equation

$$\xi(s) = \xi(1 - s), \tag{3.4}$$

while since $\xi(s)$ is real for real s, we may deduce

$$\overline{\xi(s)} = \xi(1 - \bar{s}), \tag{3.5}$$

where as usual the overbar denotes the complex conjugate. Equations (3.4) and (3.5) constitute the two basic symmetries of the Riemann xi function, such that if in the critical strip $0 < \Re(s) < 1$, (x_0, y_0) were a known zero of the zeta function $\zeta(s)$, and therefore also of the xi function $\xi(s)$, then it follows from (3.4) and (3.5) that $(1 - x_0, \ y_0)$, $(x_0, \ -y_0)$ and $(1 - x_0, \ -y_0)$ are also zeros.

In addition, from (3.4) and (3.5) we may deduce respectively

$$\begin{aligned}
u(x, y) + iv(x, y) &= u(1 - x, -y) + iv(1 - x, -y), \\
u(x, y) - iv(x, y) &= u(x, -y) + iv(x, -y),
\end{aligned}$$

and from these relations we have

$$\begin{aligned}
u(x, y) &= u(1 - x, -y) = u(x, -y) = u(1 - x, y), \\
v(x, y) &= v(1 - x, -y) = -v(x, -y) = -v(1 - x, y),
\end{aligned} \tag{3.6}$$

which are the basic symmetries of the real and imaginary parts of $\xi(s)$. From these symmetries we may extract the following results applying along $x = 1/2$ or $y = 0$, such as

$$v(1/2, y) = 0, \quad v(x, 0) = 0,$$

$$\frac{\partial u}{\partial x}\left(\frac{1}{2}, y\right) = 0, \quad \frac{\partial v}{\partial x}(x, 0) = 0,$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, y\right) = 0, \quad \frac{\partial^2 v}{\partial x^2}(x, 0) = 0, \tag{3.7}$$

and so on.

Since $\xi(s)$ is an analytical function of s, we may couple the above information with the Cauchy-Riemann equations, namely

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.8}$$

In the following section we show the X-rays of the Riemann zeta and xi functions. The thick black lines are the curves along which the imaginary part of the function vanishes. Thus, for $\xi(s)$ we have $v(x, y) = 0$, and therefore by differentiation

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0. \tag{3.9}$$

Similarly, the thin grey lines are the curves along which the real part of the function is zero, and for $\xi(s)$ this means $u(x, y) = 0$, and along these curves we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0. \tag{3.10}$$

Clearly, we are interested in those points where a black and a grey line intersect, and from (3.8)-(3.10) it is clear that at such a point the black and grey lines are orthogonal, that is the product

6

of their gradients as determined by (3.9) and (3.10) is equal to minus one.

4. X-rays for $\zeta(s)$, $\xi^*(s)$, $\xi(s)$ and $\log \xi(s)$

In order to determine the two families of curves along which either the real or the imaginary part of a complex function is zero, we employ the following algorithm for $f(s) = u(s) + iv(s)$, where $s \in \mathbb{C}$:

   (i) Calculate values for $f(s)$,

   (ii) Replace negative values with symbol '-', and positive values with '+',

   (iii) Draw a smooth curve along the boundary between the positive and negative values.

However, as Israel [6] points out "This does not work very well near singularities of the expression, self-intersections of the curve, or points where the Jacobian determinant is zero." More sophisticated techniques are outlined in Bruce et al. [2]; which involve the Newton Raphson method and vector calculus.

Figure 2 shows the X-ray for the Riemann zeta function $\zeta(s)$, which is in complete accord with that given by Arias-de-Reyna [1] and the reader is referred to this paper for a comprehensive analysis of the curves. Figure 3 shows the X-ray for the Riemann xi function $\xi^*(s)$ as defined by equation (2.1). Because $\xi^*(s)$ also satisfies the two basic symmetries (3.4) and (3.5), we see that unlike Figure 2 the curves here are symmetric, not only about the x-axis, but also about the line $x = 1/2$, which is a consequence of the functional identity (3.4). The ovals in the Figure arise from the pole of the zeta function $\zeta(s)$ at $s = 1$, and because of the functional equation, this pole is also mirrored at $s = 0$. From equation (2.6) and the definition (3.1), it is clear that the additional factor $s(s-1)$ eliminates these effects, and the X-ray for $\xi(s)$ shown in Figure 4 confirms this fact. In this figure there are no ovals, and all the curves have a similar shape. Moreover, a grey line cuts a black line only along the line $x = 1/2$, and we may confirm that these points of intersection correspond to the well known roots of the zeta function $\zeta(s)$. Thus, Figure 4 is a pictorial representation of the validity of the Riemann hypothesis.

Also from Figure 4 we observe that between each of these roots precisely one black line crosses the line $x = 1/2$. At these points of intersections of two black lines, we may confirm that the derivative $\xi'(s)$ is zero. Since $\xi(s)$ is real along $x = 1/2$, we know by Rolle's theorem that the derivative must vanish at least once between any two zeros.

In Figure 5 we present as a further illustration the X-ray of the function $\log \xi(s)$. From the equation

$$\log \xi(s) = \log \left(u^2 + v^2 \right)^{1/2} + i \tan^{-1}(v/u), \tag{4.1}$$

it is clear that the black lines shown in Figure 5 correspond precisely to those given in Figure 4, namely both families arise from $v(x, y) = 0$, while the two grey lines shown in Figure 5 arise from the equation $u^2 + v^2 = 1$. We note from the figure that the black and grey lines intersect orthogonally as we have previously anticipated.

5. Conclusions

We have continued the work of Arias-de-Reyna [1] to determine X-rays of the Riemann zeta and associated xi functions. The X-rays are the curves along which a function of a complex variable is either purely real or purely imaginary. We have shown that the Riemann xi functions, $\xi^*(s)$

defined by (2.1) and $\xi(s)$ defined by (3.1) give increasingly simplified X-rays, such that the X-ray for $\xi(s)$ involves curves only of a single profile. This is in contrast to the X-ray for $\zeta(s)$ which involves many different shaped curves, for which the zeros need to be addressed on an individual basis. The X-ray for the function $\log \xi(s)$ gives a surprising result.

REFERENCES

[1] J. ARIAS-DE-REYNA, *X-Ray of Riemann's Zeta-Function*, submitted to a journal September 13, 2002. See xxx.arxiv.org/math.NT/0309433

[2] J.W. BRUCE ET AL, *Microcomputers and mathematics*, Cambridge University Press, 1990.

[3] J.B. CONREY, *The Riemann Hypothesis*, Notices of the AMS, 50 (2003) 341-353.

[4] H.M. EDWARDS, *Riemann's Zeta Function*, Academic Press, New York, 1974.

[5] J.M. HILL, *Curve fitting, differential equations and the Riemann hypothesis*, Ramanujan Journal, 9 (2005) 357-372.

[6] R. ISRAEL, *implicitplot($x^2/y - 1, x = -1..1, y = -1..1$); (10.5.00)*, Maple User Group Answers.
email: U. Klein@Math.RWTH-Aachen.DE

[7] E.C. TITCHMARSH, *The Theory of the Riemann Zeta-function*, 2nd Edition revised by D.R. HEATH-BROWN, Oxford Science Publications, Clarendon Press, Oxford, 1986.

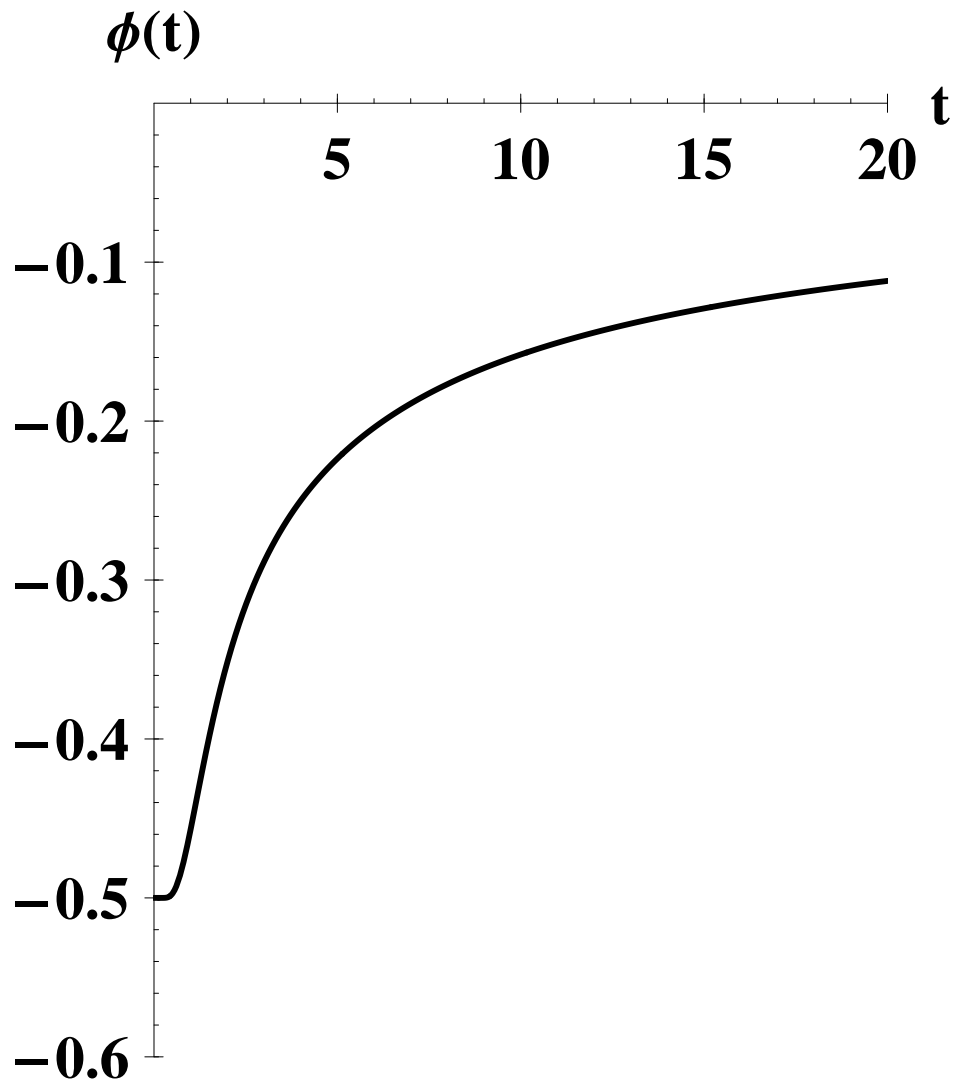[8] E.T. WHITTAKER AND G.N. WATSON, *A Course of Modern Analysis*, 4th Edition, Cambridge University Press, 1963.

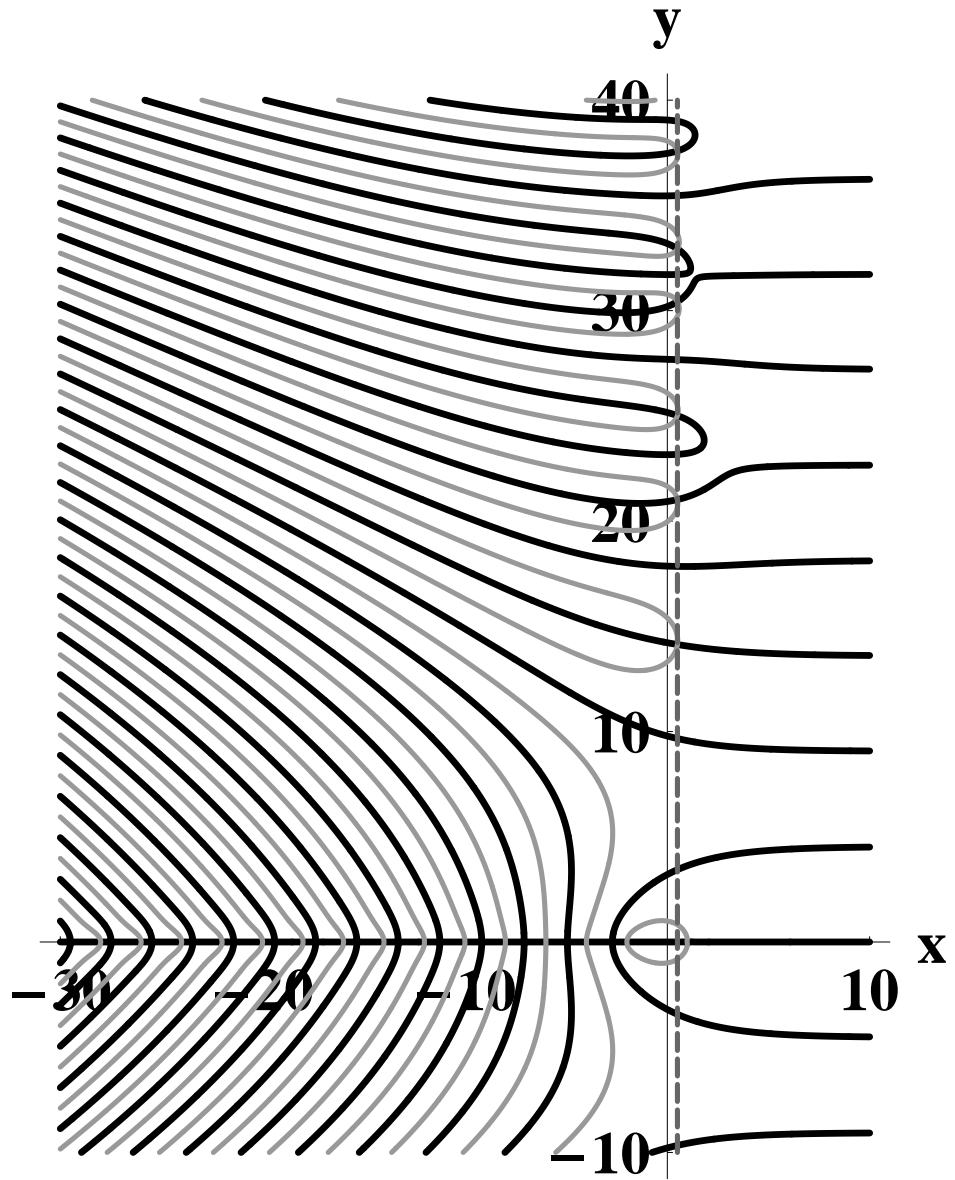Figure 1: Variation of the function $\phi(t)$ defined by (2.2).

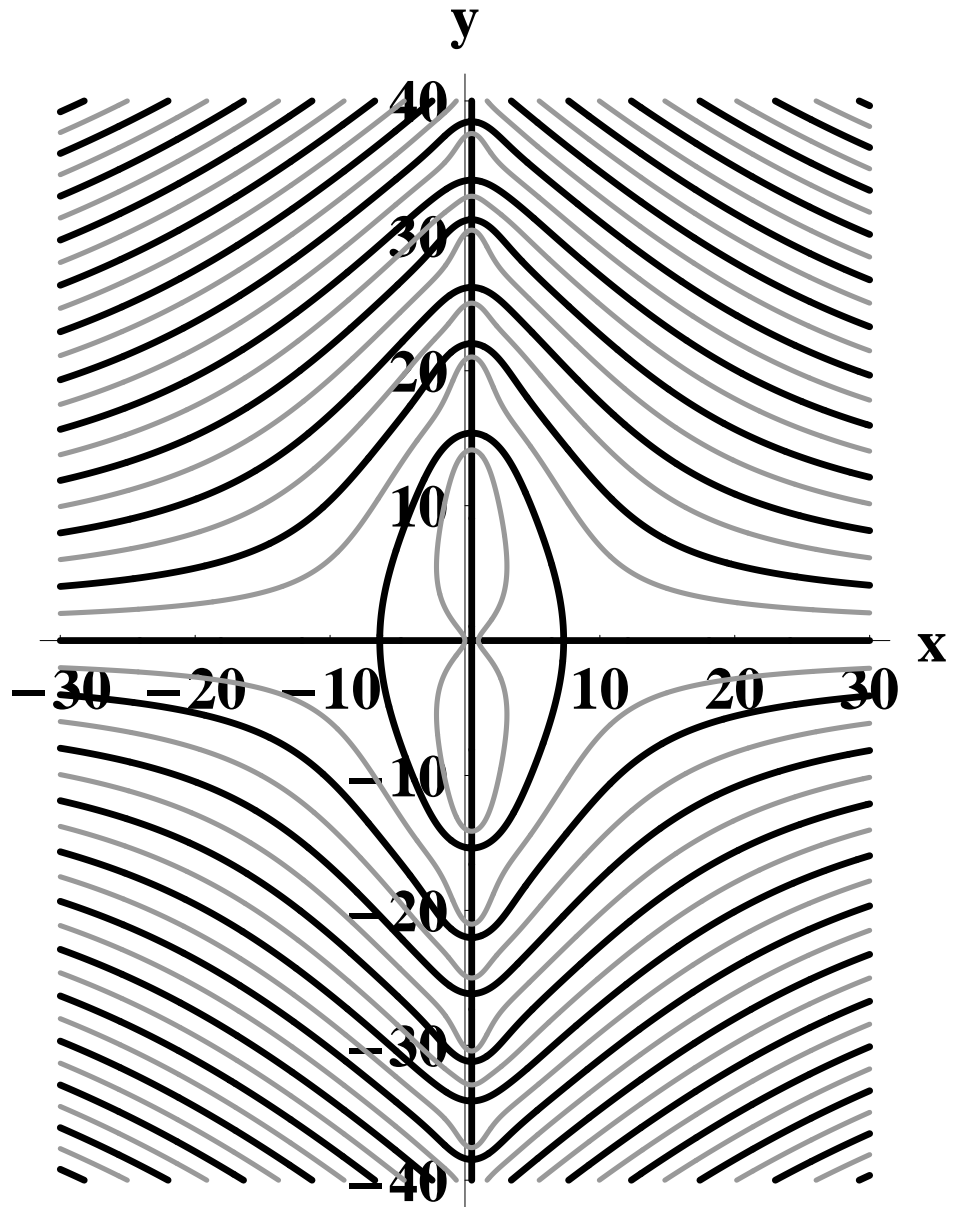Figure 2: X-ray for the Riemann zeta function $\zeta(s)$ with x=1/2 shown as $(----)$.

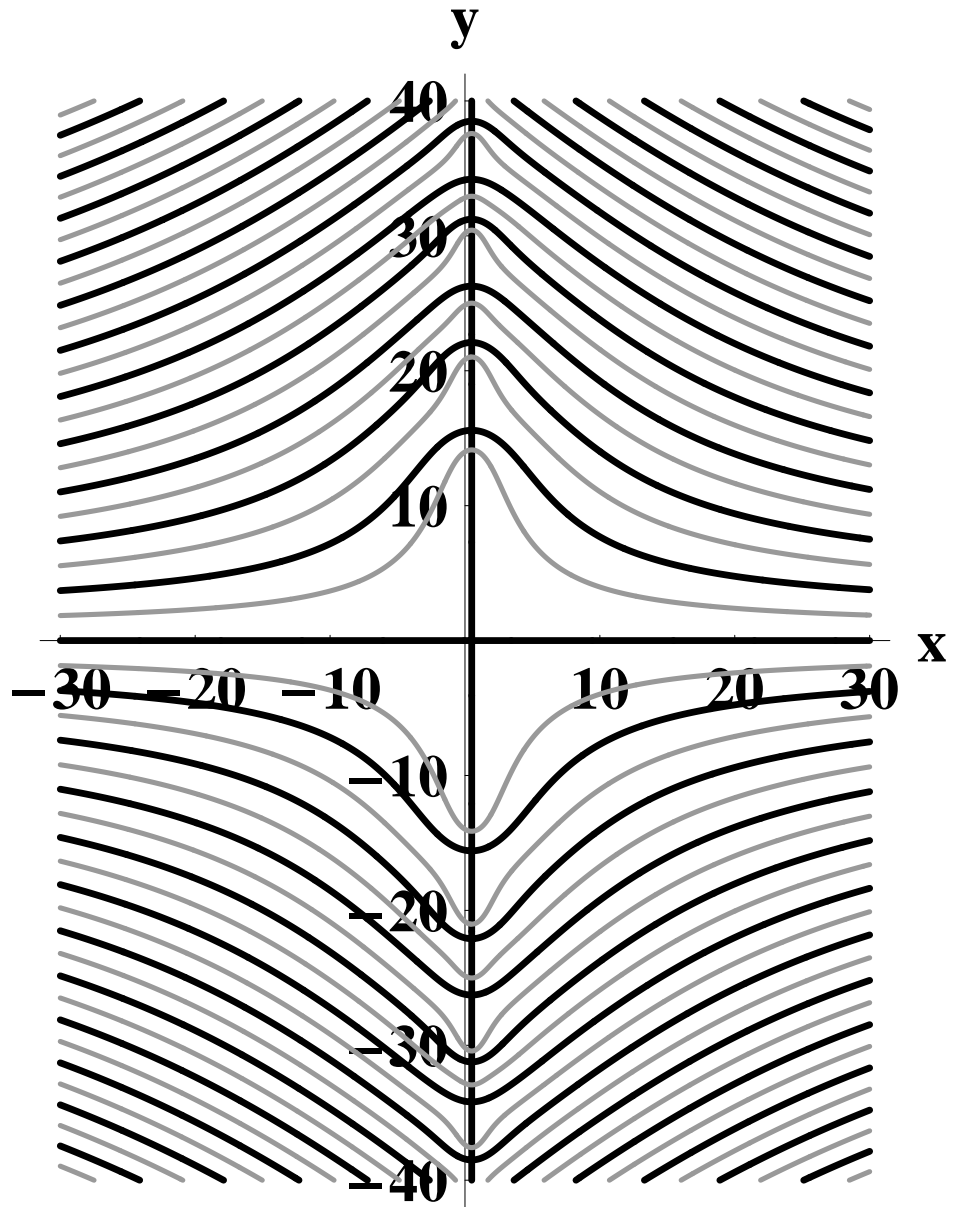Figure 3: X-ray for the Riemann xi function $\xi^*(s)$ defined by (2.1).

Figure 4: X-ray for the Riemann xi function $\xi(s)$ defined by (3.1).
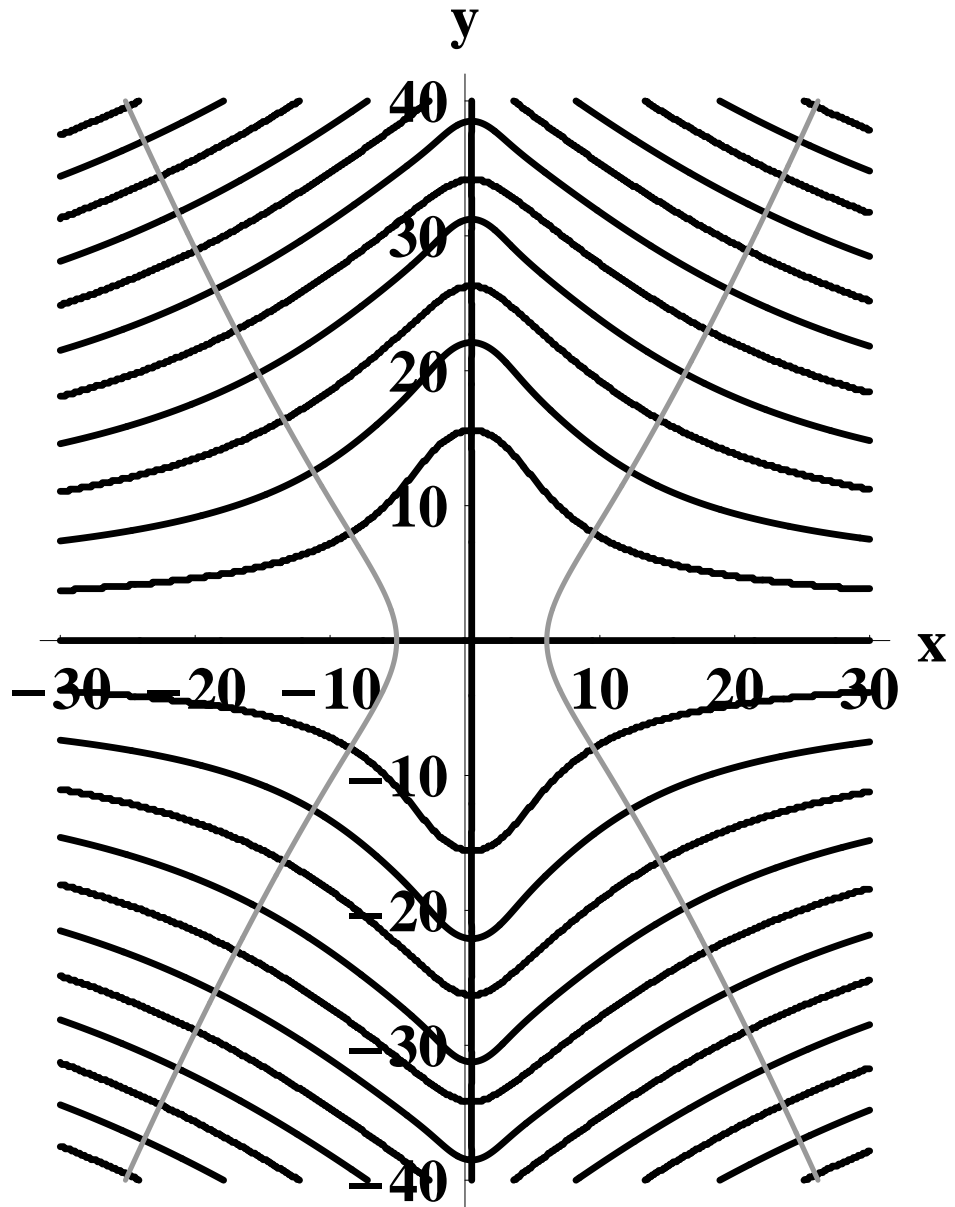
Figure 5: X-ray for the function $\log \xi(s)$, where $\xi(s)$ is defined by (3.1).