Continued Fractions in Quadratic Fields

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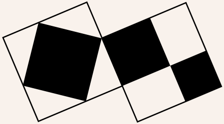

May, 2007   Algorithmic Number Theory, Turku

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The 'why this is so' of the matter is this. It happens that

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \ddots}}}}}$$

and in particular that

$$22/7 = 3 + \frac{1}{7} \quad \text{while} \quad 355/113 = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

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Simple Continued Fractions

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In general, given an irrational number α, define its sequence $(\alpha_h)_{h\geq 0}$ of complete quotients by setting $\alpha_0 = \alpha$, and $\alpha_{h+1} = 1/(\alpha_h - a_h)$. Here, the sequence $(a_h)_{h\geq 0}$ of partial quotients of α is given by $a_h = \lfloor \alpha_h \rfloor$ where $\lfloor\ \rfloor$ denotes the integer part of its argument. The truncations $[\,a_0\,,a_1\,,\dots\,,a_h\,]$ plainly are rational numbers p_h/q_h. Indeed, the continuants p_h and q_h are given by the matrix identities $h = 0, 1, 2, \dots$

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{h-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{h-1} & p_{h-2} \\ q_{h-1} & q_{h-2} \end{pmatrix}.$$

This follows readily by induction on h and the definition

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and so

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showing that the convergents do converge to a limit, namely

$$\alpha = a_0 + \sum\nolimits_{h=1}^{\infty}(-1)^{h-1}/q_{h-1}q_h\,;$$

and also that $0 < (-1)^{h-1}(\alpha - p_h/q_h) < 1/q_hq_{h+1} < 1/a_{h+1}q_h^2$. Thus, in particular

$$|\pi - 22/7| < 1/15\cdot 7^2 \quad \text{and} \quad |\pi - 355/113| < 1/292\cdot 113^2\,.$$

Conversely, suppose $q_{h-1} < q < q_h$. Because $\gcd(q_{h-1}, q_h) = 1$ there are integers a and b, with $ab < 0$, so that $q = aq_{h-1} + bq_h$. Set $p = ap_{h-1} + bp_h$. Then $q\alpha - p$ is $a(q_{h-1}\alpha - p_{h-1}) + b(q_h\alpha - p_h)$ and, since the two terms have the same sign, each must be smaller than $|q\alpha - p|$ in absolute value. Thus convergents yield locally best approximations and it follows that certainly $|q\alpha - p| > 1/2q$.

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$$p_h/q_h - p_{h-1}/q_{h-1} = (-1)^{h-1}/q_{h-1}q_h\,,$$

and so

$$p_h/q_h = a_0 + \sum_{n=1}^{h-1}(-1)^{n-1}/q_{n-1}q_n\,,$$

showing that the convergents do converge to a limit, namely

$$\alpha = a_0 + \sum\nolimits_{h=1}^{\infty}(-1)^{h-1}/q_{h-1}q_h\,;$$

and also that $0 < (-1)^{h-1}(\alpha - p_h/q_h) < 1/q_hq_{h+1} < 1/a_{h+1}q_h^2$. Thus, in particular

$$|\pi - 22/7| < 1/15\cdot 7^2 \quad \text{and} \quad |\pi - 355/113| < 1/292\cdot 113^2.$$

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The Matrix Correspondence

The principle underlying the matrix correspondence is the simple fact that a unimodular integer matrix (here, of determinant ± 1) has a decomposition as a finite product of integer matrices $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$; this product certainly is unique if all the integers are positive.

Example. By transposing the correspondence it follows that

$$[\, a_h \,, a_{h-1} \,, \ldots \,, a_1 \,] = q_h / q_{h-1}$$

$$\text{and } [\, a_h \,, a_{h-1} \,, \ldots \,, a_1 \,, a_0 \,] = p_h / q_{h-1} .$$

By the way, most of my remarks are formal: Thus, integer may be replaced by polynomial; and positive becomes of positive degree.

Question. Is it a surprise that a continued fraction expansion with partial quotients in $K[X]$ converges to a Laurent series in $K((X^{-1}))$?

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Distance

With the standard notation $p_h/q_h = [\, a_0\,, a_1\,, \ldots\,, a_h\,]$, and because $\alpha = [\, a_0\,, a_1\,, \ldots\,, a_h\,, \alpha_{h+1}\,]$, we have

$$\begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix} \begin{pmatrix} \alpha_{h+1} & 1 \\ 1 & 0 \end{pmatrix} \longleftrightarrow \frac{\alpha_{h+1}p_h + p_{h-1}}{\alpha_{h+1}q_h + q_{h-1}} = \alpha\,.$$

So, inverting the first matrix,

$$\alpha_{h+1} = -\frac{q_{h-1}\alpha - p_{h-1}}{q_h\alpha - p_h}\,.$$

The Distance Formula. It follows immediately that

$$\alpha_1\alpha_2\cdots\alpha_{h+1} = (-1)^{h+1}(p_h - \alpha q_h)^{-1}.$$

Here, I recall $p_{-1} = 1\,,\ q_{-1} = 0$. It turns out that one may usefully think of $\big|\log|p_h - \alpha q_h|\big|$ as measuring a weighted distance that the continued fraction has traversed in moving from α to α_{h+1}.

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Linear Fractional Transformations

The matrix correspondence in effect identifies 2×2 matrices $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ with linear fractional transformations $\alpha \mapsto (r\alpha + s)/(t\alpha + u)$. Thus, for arbitrary $k \neq 0$, one should identify matrices kM and M. Then any sequence $\left(\begin{smallmatrix} A_h & B_h \\ C_h & D_h \end{smallmatrix}\right)$ of nonsingular 2×2 matrices so that A_h/C_h and B_h/D_h have a common limit yields an expansion. For example, if

$$\begin{pmatrix} A_h & B_h \\ C_h & D_h \end{pmatrix} = \prod_{m=0}^{h} \begin{pmatrix} 2m+1+z & 2m+1 \\ 2m+1 & 2m+1-z \end{pmatrix},$$

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$$e - 1 = [\,1\,,1\,,2\,,1\,,1\,,4\,,1\,,1\,,6\,,\ldots\,] = [\,\overline{1\,,2h\,,1}\,]_{h=1}^{\infty}.$$

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- The fundamental property is that p/q is a continued fraction convergent of α if and only if p/q is a locally good approximation to α, roughly speaking: in the sense that $|q\alpha - p|$ is somewhat smaller than $1/q$. If so, it is locally best in that there is no rational with smaller denominator which is closer to α.
- Even the unexpected pattern in the expansion of e is, at a stretch, a corollary of the matrix correspondence.
- I add that, two numbers are equivalent if the tails of their continued fraction expansions are the same.
- We have met the distance formula

$$\alpha_1\alpha_2\cdots\alpha_{h+1} = (-1)^{h+1}(p_h - \alpha q_h)^{-1}.$$

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- Even the unexpected pattern in the expansion of e is, at a stretch, a corollary of the matrix correspondence.
- I add that, two numbers are equivalent if the tails of their continued fraction expansions are the same.
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Continued Fractions of Quadratic Irrationals

In these remarks, ω is a quadratic irrational integer of norm n and trace t; that is, $\omega^2 - t\omega + n = 0$. Because ω is a integer, both its trace $t = \omega + \overline{\omega}$ and norm $n = \omega\overline{\omega}$ must be rational integers. Because ω is irrational its discriminant $(\omega - \overline{\omega})^2$, that is $t^2 - 4n$, is not a rational square.

Further, set $\alpha := (\omega + P)/Q$ where the positive integer Q divides the norm $(\omega + P)(\overline{\omega} + P)$. This last condition is a critical convention: indeed Q dividing the norm is equivalent to the $\mathbb{Z}$-module $\langle Q, \omega + P \rangle_{\mathbb{Z}}$ being more, in fact it then is an ideal of the integral domain $\mathbb{Z}[\omega]$. To see this, it suffices to notice that

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Writing $\beta = (\sqrt{-163} + 17)/21$ is less than ideal; it is not admissible.

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In fact, $\beta = (\sqrt{-7987} + 119)/147$.

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It is quite straightforward to find the expansion of a real root of a polynomial equation. I instance this by detailing the case $f(X) = X^3 - X^2 - X - 1$. Then f has one real zero, say γ, where plainly $1 < \gamma < 2$ so $c_0 = 1$ and clearly $\gamma_1 = 1/(\gamma - c_0)$ is a zero of the polynomial $f_1(X) = -X^3 f(X^{-1} + c_0) = 2X^3 - 2X - 1$. One sees that $\lfloor \gamma_1 \rfloor = 1$, so $c_1 = 1$ and $f_2(X) = -X^3 f_1(X^{-1} + c_1)$ is given by $X^3 - 4X^2 - 6X - 2$. A little more subtly, it happens that $\lfloor \gamma_2 \rfloor = 5$ and so $f_3(X) = 7X^3 - 29X^2 - 11X - 1$ and the integer part of its real zero γ_3 is $c_3 = 4$. That yields ...

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h	c_h	$a_0^{(h)}$	$a_1^{(h)}$	$a_2^{(h)}$	$a_3^{(h)}$
0	1	1	-1	-1	-1
1	1	2	0	-2	-1
2	5	1	-4	-6	-2
3	4	7	-29	-11	-1
4	2	61	-93	-55	-7
5	305	1	-305	-273	-61
6	1	83326	-92752	-610	-1
7	8	10037	-63864	-1 57226	-83326
8	2	2 89486	-7 48054	-1 77024	-10037
9	1	10 40413	-3 04592	-9 88862	-2 89486
10	4	5 42527	-15 23193	-28 16647	-10 40413
11	6	19 56361	-110 39105	-49 87131	-5 42527
12	14	52 99117	-738 30597	-241 75393	-19 56361
13	3	2704 31827	-10244 48687	-1487 32317	-52 99117
14	1	23698 74922	-10062 34890	-14094 37756	-2704 31827
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The quadratic case is different in the critical fact that the coefficients of the f_h are bounded.

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14	1	23698 74922	-10062 34890	-14094 37756	-2704 31827
15	13	3162 29551	-36877 17230	-61033 89876	-23698 74922

But wait, there's more! Quite exceptionally, $\gamma^{-17} = 56 - 103\gamma^{-1}$.
That's the reason I chose the polynomial f.

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11	6	19 56361	-110 39105	-49 87131	-5 42527
12	14	52 99117	-738 30597	-241 75393	-19 56361
13	3	2704 31827	-10244 48687	-1487 32317	-52 99117
14	1	23698 74922	-10062 34890	-14094 37756	-2704 31827
15	13	3162 29551	-36877 17230	-61033 89876	-23698 74922

But wait, there's more! Quite exceptionally, $\gamma^{-17} = 56 - 103\gamma^{-1}$.
That's the reason I chose the polynomial f.

Note that, indeed, $y_4^3 f(x_4/y_4) = -1$.

h	c_h	x_h	y_h	$x_h^3 - x_h^2 y_h - x_h y_h^2 - y_h^3$
		0	1	
		1	0	1
0	1	1	1	-2
1	1	2	1	1
2	5	11	6	-7
3	4	46	25	61
4	2	103	56	-1
5	305	31461	17105	83326
6	1	31564	17161	-10037
7	8	2 83973	1 54393	2 89486
8	2	5 99510	3 25947	-10 40413
9	1	8 83483	4 80340	5 42527
10	4	41 33442	22 47307	-19 56361
11	6	256 84135	139 64182	52 99117
12	14	3637 11332	1977 45855	-2704 31827
13	3	11168 18131	6072 01747	23698 74922
14	1	14085 29463	8049 47602	-3162 29551
15	13			

Reduced Elements

Recall that $\alpha := (\omega + P)/Q$ where the positive integer Q divides the norm of its numerator. If ω is real, so if its discriminant $t^2 - 4n$ is positive, then I distinguish ω from its conjugate $\overline{\omega}$ by insisting that $\omega > \overline{\omega}$. One now says that α is reduced if and only if

$$\alpha > 1 \quad \text{but} \quad -1 < \overline{\alpha} < 0\,.$$

If ω is imaginary then its discriminant $t^2 - 4n$ is negative. In this case one says that α is reduced if and only if both

$$|\alpha + \overline{\alpha}| \leq 1 \quad \text{and} \quad \alpha\overline{\alpha} \geq 1\,.$$

Exercise. Confirm that if a real α is reduced then necessarily both $2P + t$ and Q are positive and less than $\omega - \overline{\omega}$.

All real quadratic irrationals have periodic continued fraction expansions. I will show that a real α has a purely periodic expansion if and only if it is reduced.

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The Continued Fraction Expansion

Write a_h for the integer part $\lfloor \alpha_h \rfloor$ of $\alpha_h = (\omega + P_h)/Q_h$; so a_h is a partial quotient in the continued fraction expansionof α_h, and the first step in that expansion is

$$\alpha_h = (\omega + P_h)/Q_h = a_h - (\overline{\omega} + P_{h+1})/Q =: a_h - \overline{\rho}_{-h}\,;$$

here $P_{h+1} := a_h Q_h - P_h - t$. Then obviously $-1 < \overline{\rho}_{-h} < 0$ because $-\overline{\rho}_{-h}$ is the fractional part of α_h. Now consider the conjugate step

$$\rho_{-h} = (\omega + P_{h+1})/Q_h = a_h - (\overline{\omega} + P_h)/Q_h = a_h - \overline{\alpha}_h\,.$$

One sees that a_h, which began life as the integer part of α_h, also is the integer part of ρ_{-h} and that also ρ_{-h} is reduced. It now follows that $\alpha_{h+1} := -1/\overline{\rho}_{-h} = (\alpha + P_{h+1})/Q_{h+1}$, the next complete quotient in the expansion, also is reduced.

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Thus the continued fraction expansion of a reduced quadratic irrational $\alpha_0 = (\omega + P_0)/Q_0$ is a sequence of steps $h = 0, 1, 2, \ldots$

$$\alpha_h = (\omega + P_h)/Q_h = a_h - (\overline{\omega} + P_{h+1})/Q_h = a_h - \overline{\rho}_{-h};$$

where $P_h + P_{h+1} + t = a_h Q_h$,

$$-Q_h Q_{h+1} = (\omega + P_{h+1})(\overline{\omega} + P_{h+1}),$$

and $\alpha_{h+1} = (\omega + P_{h+1})/Q_{h+1}$. Here all the complete quotients α_h and all the 'remainders' ρ_{-h} are reduced quadratic irrationals.

Periodicity of the expansion. Because the α_h are reduced it follows that $\omega - \overline{\omega}$ bounds both $2P_h + t$ and Q_h. Hence there are only finitely many possibilities for a step in the expansion.

Exercise. For discussion: "Finitely many" only means "fewer than infinity". But here we have much more explicit information. Explain how one might obtain a good upper bound on the length of an ideal cycle in the domain $\mathbb{Z}[\omega]$, say as a function of $D = t^2 - 4n$ as $D \to \infty$.

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		h	p_h	q_h
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$(\sqrt{46}+6)/10$	$= 1 - (-\sqrt{46}+4)/10$	1	7	1
$(\sqrt{46}+4)/3$	$= 3 - (-\sqrt{46}+5)/3$	2	27	4
$(\sqrt{46}+5)/7$	$= 1 - (-\sqrt{46}+2)/7$	3	34	5
$(\sqrt{46}+2)/6$	$= 1 - (-\sqrt{46}+4)/6$	4	61	9
$(\sqrt{46}+4)/5$	$= 2 - (-\sqrt{46}+6)/5$	5	156	23
$(\sqrt{46}+6)/2$	$= 6 - (-\sqrt{46}+6)/2$	6	997	147
$(\sqrt{46}+6)/5$	$= 2 - (-\sqrt{46}+4)/5$	7	2150	317
$(\sqrt{46}+4)/6$	$= 1 - (-\sqrt{46}+2)/6$	8	3147	464
$(\sqrt{46}+2)/7$	$= 1 - (-\sqrt{46}+5)/7$	9	5297	781
$(\sqrt{46}+5)/3$	$= 3 - (-\sqrt{46}+4)/3$	10	19038	2807
$(\sqrt{46}+4)/10$	$= 1 - (-\sqrt{46}+6)/10$	11	24335	3588
$(\sqrt{46}+6)/1$	$= 12 - (-\sqrt{46}+6)/1$	12		

Here we see $\omega = \sqrt{46}$ displaying its period of length $r = 12$.

In particular, $24335^2 - 46 \cdot 3588^2 = 1$.

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Here we see $\omega = \sqrt{46}$ displaying its period of length $r = 12$. The convergents p_h/q_h also computed here provide interesting identities $p_h^2 - 46q_h^2 = (-1)^{h+1}Q_{h+1}$. In particular, $24335^2 - 46 \cdot 3588^2 = 1$.

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Summary of Continued Fractions of Algebraic Numbers

- There is a fine algorithm for computing the continued fraction expansion of any algebraic number. However, the quadratic case is particularly good because only finitely many different complete quotients can occur; so the expansion is eventually periodic.
- I deal with an arbitrary real irrational quadratic integer ω but, in truth, I intend primarily the two cases $\omega = \sqrt{D}$ with $n = -D$ and $t = 0$, so $\Delta = t^2 - 4n = 4D$; and, provided that D is $1 \bmod 4$, $\omega = \frac{1}{2}(1 + \sqrt{D})$, with $n = \frac{1}{4}(1 - D)$ and $t = 1$, so $\Delta = D$.
- Here D is a positive integer, not a square. Actually, it's psychologically good always to take D to be a discriminant, so 0 or $1 \bmod 4$; then the basic choices for ω are $\frac{1}{2}\sqrt{D}$ or $\frac{1}{2}(1 + \sqrt{D})$ according to the parity of D. Now the discriminant always is D.
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A Useful Conjugation

Suppose then that step $r-1$ is the first step in the tableau to coincide with an earlier step. Then the period length of the expansion of α is at most r and, unless step $r-1$ happens to coincide with step 0, the expansion will have a pre-period.

However, consider the continued fraction expansion of ρ_{-r+1}, recalling that it commences with the step

$$(\omega + P_r)/Q_{r-1} = a_{r-1} - (\overline{\omega} + P_{r-1})/Q_{r-1} = a_{r-1} - \overline{\alpha}_{r-1}\,.$$

Because this expansion is the conjugate of the continued fraction expansion of α it too must have a period of length at most r. Because it commences with the conjugate of the first repeated line and runs in the direction opposite to that of the expansion of α, it must be purely periodic. But any putative pre-period of α would provide a post-period for ρ_{-r+1}; which is absurd. So also the expansion of α is purely periodic.

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Suppose then that step $r-1$ is the first step in the tableau to coincide with an earlier step. Then the period length of the expansion of α is at most r and, unless step $r-1$ happens to coincide with step 0, the expansion will have a pre-period.

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and is symmetric, that is unchanged under conjugation. Though it is more natural to expand ω rather than $\omega + A - t$, I choose the latter because, unlike ω, it certainly is reduced.

Exercise. (a) Observe in the case $\omega + A - t$ that the period must have a second symmetry (at any rate, if $r > 1$). Moreover, if $r = 2k$ is even then this symmetry is given by $\alpha_k = \rho_{-k}$, and if $r = 2k + 1$ is odd then $\rho_{-k+1} = \alpha_k$. (b) It has been compellingly put to me that "Mathematics is the study of degeneracy". The degenerate case here is $r = 1$. Does claim (a) remain true in essence (as it certainly should) for $r = 1$? (c) It is not true that every α has a symmetric period. Comment on the claim that the period of α has symmetries if and only if either (i) there is an h so that α_h has integral trace or (ii) so that $\alpha_h \overline{\alpha}_h = -1$. (d) Give examples illustrating the various claims just now made.

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Vincent (1836) reports that either the γ_h all lie in the left hand half of the unit circle once h is sufficiently large, or $\gamma = \alpha$ and they all are greater than 1. So what?

Suppose α is a real quadratic irrational and consider the α_h, recalling complete quotients all are greater than 1. But their conjugates $\overline{\alpha_h}$ are the result of $\overline{\alpha}$ having suffered the ignominy of being α-expanded. Hence, once h is large enough, they all satisfy $-1 < \overline{\alpha_h} < 0$. In other words, the continued fraction process eventually reduces any real quadratic irrational.

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The Dirichlet Box Principle

Let a and b be be positive integers. Then $a > b$ means that if each of a objects is placed in one of b boxes then there will be at least one box containing more than one object. Accordingly, take the $Q+1$ numbers $\{0, \alpha, 2\alpha, \ldots, Q\alpha\}$, divide the unit interval into Q half-open intervals $[(i-1)/Q, i/Q[$, and place jQ into the i-th interval if its fractional part falls into that interval. Then there will be at least one interval containing two of the numbers, proving that there is a positive integer $q \leq Q$ so that the distance $\|q\alpha\|$ of $q\alpha$ to its nearest integer satisfies

$$\|q\alpha\| < 1/Q; \quad \text{say } |q\alpha - p| < 1/q, \text{ some integer } 0 < q \leq Q.$$

I next apply the box principle and its useful corollary to showing that real quadratic domains $\mathbb{Z}[\omega]$ contain non-trivial units, to wit elements different from ± 1, yet dividing 1. The periodicity of the continued fraction expansion of a real quadratic irrational is a corollary. The argument is independent of our earlier one.

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Let a and b be be positive integers. Then $a > b$ means that if each of a objects is placed in one of b boxes then there will be at least one box containing more than one object. Accordingly, take the $Q+1$ numbers $\{0, \alpha, 2\alpha, \ldots, Q\alpha\}$, divide the unit interval into Q half-open intervals $[(i-1)/Q, i/Q[$, and place jQ into the i-th interval if its fractional part falls into that interval. Then there will be at least one interval containing two of the numbers, proving that there is a positive integer $q \le Q$ so that the distance $\|q\alpha\|$ of $q\alpha$ to its nearest integer satisfies

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Units in Quadratic Orders

Given ω, it follows from Dirichlet's argument that there are infinitely many integers q so that $\|q\omega\| = |q\omega - p| < 1/q$; whence, after multiplying and because $|\omega - p/q| < 1$, indeed so that $|(q\omega - p)(q\overline{\omega} - p)| < (\omega - \overline{\omega}) + 1$.

Again by the box principle, it follows that there is some integer k (with $|k| < (\omega - \overline{\omega}) + 1$) for which there are are infinitely many pairs of integers (p, q) so that $|(q\omega - p)(q\overline{\omega} - p)| = k$.

Yet again, it follows by the box principle that there is a pair of those pairs so that $p \equiv p'$ and $q \equiv q' \pmod{k}$.

Then
$$\frac{(q\omega - p)(q\overline{\omega} - p)}{(q'\omega - p')(q'\overline{\omega} - p')} = (x - \omega y)(x - \overline{\omega} y) = \pm 1$$
displays a unit $x - \omega y$; here x and y are rational integers given by $x = (pp' - tpq' + nqq')/k$ and $y = (pq' - p'q)/k$.

Exercise. Verify (or correct) all these many remarks.

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The Matrix Correspondence: *RL*-Sequences

It is often convenient to set $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, whence

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} = R^a J = J L^a .$$

Thus a continued fraction expansion $[\,a_0\,, a_1\,, a_2\,, \ldots\,]$ corresponds to an <u>*RL*-sequence</u> $R^{a_0} L^{a_1} R^{a_2} L^{a_3} R^{a_4} \ldots$. It follows, for example, that a zero partial quotient is readily dealt with by the rule

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Now let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, and $A' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Multiplying a continued fraction by 2 is the same as multiplying its *RL*-sequence on the left by A. But to turn that product back into an *RL*-sequence we now need rules for commuting the A through the sequence . . . .

Exercise. (a) Verify that $AR = R^2A$, $ALR = RLA'$, and $AL^2 = LA$; and obtain the corresponding transition rules for A'. (b) Define ω by $\omega^2 - \omega - 15 = 0$. Compute its cfe, and thence that of $\sqrt{61}$.

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Units and Periodicity

Given that $x - \omega y$ is a unit, the matrix $N = \begin{pmatrix} x & -ny \\ y & x-ty \end{pmatrix}$ has determinant ± 1 and hence decomposes as a product

$$N = \begin{pmatrix} w_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} w_r & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

there is a concluding zero because $-ny > x$.

Theorem. *The continued fraction expansion of ω is given by*

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$$\gamma \longrightarrow N \begin{pmatrix} \gamma & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma x - ny & x \\ \gamma y + x - ty & y \end{pmatrix} \longrightarrow \frac{\gamma x - ny}{\gamma y + x - ty}.$$

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Given that $x - \omega y$ is a unit, the matrix $N = \begin{pmatrix} x & -ny \\ y & x-ty \end{pmatrix}$ has determinant ± 1 and hence decomposes as a product

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there is a concluding zero because $-ny > x$.

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				0	1	
				1	0	1
0	1	0	7	7	1	-13
7	13	1	1	8	1	2
6	2	2	6	55	7	-13
6	13	3	1	63	8	1
7	1	4	14	937	119	-13
7	13	5	1	1000	127	2
6	2	6	6	6937	881	-13
6	13	7	1	7937	1008	1
7	1	8	14	118055	14993	-13
7	13	9	1	125992	16001	2
6	2	10	6	874007	110999	-13
6	13	11	1	999999	127000	1
7	1	12	14	14873993	1888999	-13
7	13	13	1	15873992	2015999	2

Here $\omega = \sqrt{62}$ and I display only the necessary data. We see that $\omega = [\overline{7, 1, 6, 1, 14}\,]$ and observe the fundamental unit $\eta = 63 - 8\omega$, and its powers $\eta^2 = 7937 - 1008\omega$, $\eta^3 = 999999 - 127000\omega$.

Exercise. For discussion. Notice that $\alpha = 8 - \omega$ has norm 2 and plainly $\alpha^2 = 2\eta$. But $7 - \omega$ has norm -13, yet . . . .

P_h	Q_h	h	a_h	x_h	y_h	$x_h^2 - 62y_h^2$
				0	1	
				1	0	1
0	1	0	7	7	1	-13
7	13	1	1	8	1	2
6	2	2	6	55	7	-13
6	13	3	1	63	8	1
7	1	4	14	937	119	-13
7	13	5	1	1000	127	2
6	2	6	6	6937	881	-13
6	13	7	1	7937	1008	1
7	1	8	14	118055	14993	-13
7	13	9	1	125992	16001	2
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	h	x_h	y_h
$(\sqrt{1891}+0)/1 = 43-(-\sqrt{1891}+43)/1$	0	43	1
$(\sqrt{1891}+43)/42 = 2-(-\sqrt{1891}+41)/42$	1	87	2
$(\sqrt{1891}+41)/5 = 16-(-\sqrt{1891}+39)/5$	2	1435	33
$(\sqrt{1891}+39)/74 = 1-(-\sqrt{1891}+35)/74$	3	1522	35
$(\sqrt{1891}+35)/9 = 8-(-\sqrt{1891}+37)/9$	4	13611	313
$(\sqrt{1891}+37)/58 = 1-(-\sqrt{1891}+21)/58$	5	15133	348
$(\sqrt{1891}+21)/25 = 2-(-\sqrt{1891}+29)/25$	6	43877	1009
$(\sqrt{1891}+29)/42 = 1-(-\sqrt{1891}+13)/42$	7	59010	1357
$(\sqrt{1891}+13)/41 = 1-(-\sqrt{1891}+28)/41$	8	102887	2366
$(\sqrt{1891}+28)/27 = 2-(-\sqrt{1891}+26)/27$	9	264784	6089
$(\sqrt{1891}+26)/45 = 1-(-\sqrt{1891}+19)/45$	10	367671	8455
$(\sqrt{1891}+19)/34 = 1-(-\sqrt{1891}+15)/34$	11	632455	14544
$(\sqrt{1891}+15)/49 = 1-(-\sqrt{1891}+34)/49$	12	1000126	22999
$(\sqrt{1891}+34)/15 = 5-(-\sqrt{1891}+41)/15$	13	5633085	129539
$(\sqrt{1891}+41)/14 = 6-(-\sqrt{1891}+43)/14$	14	34798636	800233
$(\sqrt{1891}+43)/3 = 28-(-\sqrt{1891}+41)/3$	15	979994893	22536063
$(\sqrt{1891}+41)/70 = 1-(-\sqrt{1891}+29)/70$	16	1014793529	23336296
$(\sqrt{1891}+29)/15 = 4-(-\sqrt{1891}+31)/15$	17	5039169009	115881247
$(\sqrt{1891}+31)/62 = 1-(-\sqrt{1891}+31)/62$	18	6053962538	139217543
$(\sqrt{1891}+31)/15 = 4-(-\sqrt{1891}+29)/15$	19	29255019161	672751419
$(\sqrt{1891}+29)/70 = 1-(-\sqrt{1891}+41)/70$	20	35308981699	811968962
$(\sqrt{1891}+41)/3 = \ldots$			

Ideal Matrices

Consider integer matrices of the shape $N = \begin{pmatrix} x & -ny \\ y & x-ty \end{pmatrix}$. Suppose that x and y are relatively prime, that is $\gcd(x, y) = 1$, and $\det N = \pm Q$, with $Q > 0$. Then N has a decomposition

$$N = \begin{pmatrix} x & -ny \\ y & x - ty \end{pmatrix} = \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \begin{pmatrix} 1 & P \\ 0 & Q \end{pmatrix}$$

with integers x', y' so that $xy' - x'y = \pm 1$ and some integer $P \in [0, Q[$. In brief, the decomposition provides a correspondence between N and an ideal $\langle Q, \omega + P \rangle_{\mathbb{Z}}$ of $\mathbb{Z}[\omega]$ and, this is the point, this correspondence preserves multiplication variously of the matrices and of the ideals.

Remark. We identify matrices kM and M for nonzero constants k; therefore, when multiplying matrices (or ideals) the relevant product is the one after removal of any common factor of all the elements.

Exercise. (a) Show that if Q is squarefree then it divides the matrix N^2 if and only if Q divides the discriminant $D = t^2 - 4n$. (b) Show that if $Q = 4$ then 8 divides the matrix N^3.

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$$\begin{pmatrix} x_h & -ny_h \\ y_h & x_h - ty_h \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} a_h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & P_{h+1} \\ 0 & Q_{h+1} \end{pmatrix}.$$

Exercise. (a) Show that the product of any two ideal matrices is indeed again a matrix of that special shape. (b) Explain why that is obvious from the word 'go' without a laboured multiplication.

Summary of Period Cycles

- We have seen several proofs that a reduced element is part of a period, or cycle, of equivalent reduced elements. Because elements $(\omega + P)/Q$ correspond to $\mathbb{Z}[\omega]$-ideals $\langle Q, \omega + P\rangle$ we may equally speak of ideal cycles.
- Moreover, a cycle provides a (nontrivial) unit in $\mathbb{Z}[\omega]$; conversely a unit induces a cycle.
- The distance formula entails that the fundamental unit, say $x - \omega y$, provides the length $-\log|x - \omega y|$ of the cycle. This quantity is also known as the regulator of $\mathbb{Z}[\omega]$.
- Roughly, this length is $\log r$; where r is the number of steps of the period. However, r is usually quite large, $\sqrt{D}\log\log D$ or so. Hence, for serious D, units are mostly enormous, typically so big that it is totally infeasible to display them in any naïve way.
- In brief, in practice one cannot detail the continuants x_h and y_h. The ideal matrices truly are "ideal", but only in the sense "unreal" or "theoretical".

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Continued Fractions in Function Fields

We suppose integer $\longleftarrow$ polynomial, and positive $\longleftarrow$ of positive degree. To fix matters we suppose the polynomials to be defined over some base field K and remark that K may be infinite or finite. A useful analogue for the real numbers is provided by the field of Laurent series $K((X^{-1}))$, instanced by

$$F(X) = \sum_{h=-m}^{\infty} f_{-h} X^{-h} .$$

The example series F has degree m and its integer part is the polynomial $\lfloor F \rfloor = f_m X^m + f_{m-1} X^{m-1} + \cdots + f_1 X + f_0$.

Matters are exactly as or more simple than in the numerical case. Convergents are quotients of relatively prime polynomials, continued fractions converge to Laurent series; but x/y is a convergent of F if and only if $\deg(x - Fy) < -\deg y$. One point that needs care is, however, that the non-zero elements of K all are (trivial) units of $K[X]$; this fact has some seemingly nontrivial consequences.

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Multlipying a Continued Fraction by a Constant

Multiplying a continued fraction $[\,a_0\,,a_1\,,a_2\,,a_3\,,\ldots\,]$ by x leads to $[\,xa_0\,,a_1/x\,,xa_2\,,a_3/x\,,\ldots\,]$, with the partial quotients alternately being multiplied and divided. An elegant version of the rule is given by

$$x[\,ya_0\,,xa_1\,,ya_2\,,xa_3\,,ya_4\,,\ldots\,]=y[\,xa_0\,,ya_1\,,xa_2\,,ya_3\,,xa_4\,,\ldots\,].$$

Obviously, unless the multiplier is a unit, in general multiplication (or division) leads to drastically inadmissible partial quotients seriously polluting the expansion. There are tricks whereby one readies an expansion for the multiplication, as in the 'elegant version' above. Or, there is a fine algorithm of George Raney viewing the multiplication as a multiple state transduction of an RL-sequence.

Even when the multiplication is by a unit, so that no great harm is done, the effect on the expansion may be startling and unexpected. In the case of quadratic irrationals over function fields, it creates the possibility of quasi-periodicity, where a 'wannabe' period in fact presents as a sequence of multiples of itself by k, k^2, k^3, ...

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Continued Fraction of the Square Root of a Polynomial

Set $Y^2 = D(X)$ where $D \neq \square$ is a monic polynomial of degree $2g+2$.
Then we may write

$$D(X) = \big(A(X)\big)^2 + 4R(X)\,,$$

where A is the polynomial part of the square root Y of D and $4R$, with $\deg R \le g$, is the remainder. We then take

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thereby viewing Y as an element of $K((X^{-1}))$, Laurent series in the variable $1/X$. All this makes sense over any base field K not of characteristic 2.

However, below I deal with the quadratic irrational function Z defined by

$$\mathcal{C} : Z^2 - AZ - R = 0\,; \quad \text{in effect } Z = \tfrac{1}{2}(Y + A). \qquad (\ddagger)$$

Then $\deg Z = \deg A = g+1$, while its conjugate satisfies $\deg \overline{Z} < 0$; so Z is reduced. Note that Z makes sense in arbitrary characteristic, including characteristic two.

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Exercise. Confirm that (a) given that Z is in $K((X^{-1}))$, there plainly is no loss of generality in supposing, as I have, that $\deg R < \deg A$, equivalently that A is the polynomial part of $\sqrt{D} = Z - \overline{Z}$, the square root of the discriminant $D = A^2 + 4A$ of Z; and (b) given that $\deg Z > \deg \overline{Z}$, the conditions $\deg Z > 0$ and $\deg \overline{Z} < 0$ precisely affirm that Z is reduced, in the sense that the continued fraction process on a quadratic irrational always leads to and then sustains the conditions.

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Exercise. Confirm that (a) given that Z is in $K((X^{-1}))$, there plainly is no loss of generality in supposing, as I have, that $\deg R < \deg A$, equivalently that A is the polynomial part of $\sqrt{D} = Z - \overline{Z}$, the square root of the discriminant $D = A^2 + 4A$ of Z; and (b) given that $\deg Z > \deg \overline{Z}$, the conditions $\deg Z > 0$ and $\deg \overline{Z} < 0$ precisely affirm that Z is reduced, in the sense that the continued fraction process on a quadratic irrational always leads to and then sustains the conditions.

Technically, Z is a real quadratic irrational function; quadratic irrationals defined over $K[X]$ but not in $K((X^{-1}))$ are imaginary.

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Periodicity of Continued Fraction Expansions

Set $Z_h = (Z + P_h)/Q_h$ where P_h and Q_h are polynomials with Q_h dividing the norm $(Z + P_h)(\overline{Z} + P_h)$. Suppose that $\deg Z_h > 0$ and $\deg \overline{Z}_h < 0$; in other words that Z_h is reduced.

Exercise (a) Show that Z_h is reduced if and only if $\deg P \leq g - 1$ and $\deg Q_h \leq g$. (b) Denote the polynomial part of Z_h by a_h and set $Z_h = a_h - \overline{R}_{-h}$. Parody the argument of the numerical case to confirm that R_{-h} and $Z_{h+1} = -1/\overline{R}_{-h}$ are reduced.

It seems to follow that every reduced element must have a purely periodic continued fraction expansion. And that's true, but only sort of. The trouble is that if the base field K is infinite then the period is generically of infinite length. The point is that the box principle does not apply because if K is infinite then there are infinitely many polynomials of bounded degree.

More, it is then rare and unusual happenstance for any reduced Z_0 to have a periodic expansion.

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Continued Fractions and Hyperelliptic Curves

Note that $(Z+P_h)(\overline{Z}+P_h) = -R + AP_h + P_h^2$. Suppose ϑ_h denotes a typical zero of Q_h. Then the condition: Q_h divides the norm $(Z+P_h)(\overline{Z}+P_h)$ asserts that $R(\vartheta_h) = \big(A(\vartheta_h) + P_h(\vartheta_h)\big)\, P_h(\vartheta_h)$.

However, recall that Z defines the hyperelliptic curve

$$\mathcal{C} : Z^2 - AZ - R = 0$$

of genus g. Of course $\mathcal{C}$ is defined over K but, for a moment disregarding that, it follows from the remarks above that the point $\big(\vartheta_h, -P_h(\vartheta_h)\big)$ is a point on $\mathcal{C}$. In general $\deg Q_h = g$ and so has g conjugate zeros. That gives a g-tuple of conjugate points on $\mathcal{C}$, or in proper language, a divisor defined over K on $\mathcal{C}$.

Equivalence classes of divisors provide the points of the Jacobian of $\mathcal{C}$. So the continued fraction provides a sequence of points on $\mathrm{Jac}(\mathcal{C})$. It turns out that consecutive such points differ by some multiple (in fact the degree of a_h) of the class of the divisor at infinity on $\mathcal{C}$.

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Negative Continued Fraction Expansions

We get the sequence of positive partial quotients, say (a_h), of a simple continued fraction expansion by underestimating each successive complete quotient by its floor. We obtain

$$[\, a_0\,, a_1\,, a_2\,, \ldots\,] = a_0 + \frac{1}{a_1\ +}\ \frac{1}{a_2\ +}\ \frac{1}{a_3\ +}\ \frac{1}{a_4\ +}\ \cdots$$

If, instead, we define the partial quotients by overestimating the successive complete quotients by their ceiling, we obtain a negative continued fraction with partial quotients (b_h), say. But a negative continued fraction is just a regular continued fraction with partial quotients of alternating sign:

$$[\, b_0\,, b_1\,, b_2\,, \ldots\,]^- = b_0 - \frac{1}{b_1\ -}\ \frac{1}{b_2\ -}\ \frac{1}{b_3\ -}\ \frac{1}{b_4\ -}\ \cdots$$

$$= b_0 + \frac{1}{\overline{b}_1\ +}\ \frac{1}{b_2\ +}\ \frac{1}{\overline{b}_3\ +}\ \frac{1}{b_4\ +}\ \cdots = [\, b_0\,, \overline{b}_1\,, b_2\,, \overline{b}_3\,, b_4\,, \overline{b}_5\,, \ldots\,].$$

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Of course, formulas for evaluating continued fractions cannot know or care about the signs of partial quotients. If one is so moved, one may and can change the sign of all succeeding partial quotients in an expansion by inserting the string $0,\overline{1},1,\overline{1},0$ into an expansion. Then, for example,

$$-\pi = [\overline{3},\overline{7},\overline{15},\overline{1},\overline{292},\overline{1},\overline{\ldots}]$$
$$= [\overline{3},0,\overline{1},1,\overline{1},0,7,15,1,292,1,\ldots]$$
$$= [\overline{4},1,6,15,1,292,1,\ldots].$$

Negation Lemma. The computation

$$-\beta = 0 + \overline{\beta}$$
$$-1/\beta = \overline{1} + (\beta - 1)/\beta$$
$$\beta/(\beta - 1) = 1 + 1/(\beta - 1)$$
$$\beta - 1 = \overline{1} + \beta$$
$$1/\beta = 0 + 1/\beta$$
$$\beta = \cdots$$

shows that $-\beta = [0,\overline{1},1,\overline{1},0,\beta]$.

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(b) Does the Negation Lemma above fully justify my insertion claim?

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All this is enough to provide a succinct summary of just how a simple continued fraction expansion $[\,a_0\,,a_1\,,a_2\,,\ldots\,]$ — thus with all the a_h positive, may be transformed into a negative continued fraction $[\,b_0\,,\overline{b}_1\,,b_2\,,\overline{b}_3\,,b_4\,,\ldots\,]$ — where the entries have alternating sign. In brief, one arranges the alternation of sign by alternately inserting the appropriate word $0\,,\overline{1}\,,1\,,\overline{1}\,,0$ or $0\,,1\,,\overline{1}\,,1\,,0$ between the first pair of consecutive partial quotients that still have the same sign. One finds that $[\,a_0\,,a_1\,,a_2\,,\ldots\,]$ becomes the negative continued fraction

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An Astonishing Result

Set $\omega = \sqrt{p}$ where $p \equiv 3 \pmod 4$ is a prime number other than 3 with the property that $\mathbb{Q}(\omega)$ has class number $h(p) = 1$ (that is, the reduced elements of $\mathbb{Q}(\omega)$ make up just one cycle). Then $\frac{1}{3}(b_0 + b_1 + \cdots + b_{r-1}) - r$ is the number $h(-p)$ of distinct equivalence classes of quadratic forms of discriminant $-p$; here $b_0, b_1, \ldots, b_{r-1}$ is the (minimal) period of the negative continued fraction expansion of $\sqrt{p} + \lceil\sqrt{p}\rceil$.

Even if one does not at all understand what the theorem alleges, the incidental implication that the sum $b_0 + b_1 + \cdots + b_{r-1}$ must be divisible by 3 should astonish. Note that experimentally and conjecturally a majority of primes $p = 4n + 3$ have class number 1.

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An Astonishing Result

Set $\omega = \sqrt{p}$ *where* $p \equiv 3 \pmod 4$ *is a prime number other than* 3 *with the property that* $\mathbb{Q}(\omega)$ *has class number* $h(p) = 1$ *(that is, the reduced elements of* $\mathbb{Q}(\omega)$ *make up just* one *cycle). Then* $\frac{1}{3}(b_0 + b_1 + \cdots + b_{r-1}) - r$ *is the number* $h(-p)$ *of distinct equivalence classes of quadratic forms of discriminant* $-p$*; here* $b_0, b_1, \ldots, b_{r-1}$ *is the (minimal) period of the negative continued fraction expansion of* $\sqrt{p} + \lceil\sqrt{p}\rceil$.

Even if one does not at all understand what the theorem alleges, the incidental implication that the sum $b_0 + b_1 + \cdots + b_{r-1}$ must be divisible by 3 should astonish. Note that experimentally and conjecturally a majority of primes $p = 4n + 3$ have class number 1.

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A reasonably hefty example may be helpful. Take $p = 163$ and set $\omega = \sqrt{163}$; note that $\lfloor \omega \rfloor = 12$. Then

$$\begin{aligned}
(\omega + 12)/1 &= 24 - (\omega + 12)/1 \\
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So $\omega + 12 = [\overline{24, 1, 3, 3, 2, 1, 1, 7, 1, 11, 1, 7, 1, 1, 2, 3, 3, 1}]$.

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Some 163 wonders. The polynomial $f(x) = x^2 + x + 41$ has the interesting property that $f(0) = 41$, $f(1) = 43$, $f(2) = 47$, $f(3) = 53$, $f(4) = 61$, $f(5) = 71$, $f(6) = 83$, $f(7) = 97$, $f(8) = 113$, $f(9) = 131$, $f(10) = 151$, $\ldots$, with all those values prime.

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Short Periods

The examples $\omega = \sqrt{W^2+1}$ trivially provide

$$\omega + |W| = 2|W| - (\overline{\omega} + |W|)\,,$$

displaying period length 1.

Exercise. (a) Notice here that $n = \omega\overline{\omega} = -1$. Comment. (b) Is it obvious, or even true, that the example gives *all* cases of period length 1?

It turns out that the correct generalisation of our examples is the cases $\sqrt{W^2+c}$ with c dividing $4W$. I make the divisibility manifest by considering the cases $\sqrt{a^2W^2+4a}$.

Suppose we ask much more generally for polynomials $F = F(W)$ so that, as W varies in $\mathbb{Z}$, (i) $F(W)$ takes only integer values not all square and (ii) the period length of the continued fraction expansion of $\sqrt{|F(W)|}$ is bounded independent of W (thus in terms of F alone).

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Suppose we ask much more generally for polynomials $F = F(W)$ so that, as W varies in $\mathbb{Z}$, (i) $F(W)$ takes only integer values not all square and (ii) the period length of the continued fraction expansion of $\sqrt{|F(W)|}$ is bounded independent of W (thus in terms of F alone).

These questions, specifically (ii), were ingeniously asked and fully answered by Andrzej Schinzel more than forty years ago[†]. In particular, if F is of odd degree or if its leading coefficient is not a square then the periods certainly are unbounded, so I presume from here on that F has even degree and has square leading coefficient.

In that case, the period is bounded if and only if,

1. $Y = \sqrt{F(X)}$ has a periodic continued fraction expansion as a quadratic irrational integral function in the domain $\mathbb{Q}[X, Y]$ — such expansions are only periodic by happenstance, because $\mathbb{Q}$ is infinite; and
2. some resulting nontrivial unit of norm dividing 4 in the quadratic function field $\mathbb{Q}(X, Y)$ must have its coefficients in $\mathbb{Z}$, that is, it must be an element of $\mathbb{Z}[X, Y]$.

Roger Patterson and I have called this second criterion Schinzel's Condition. For F quadratic only Schinzel's Condition is relevant.

[†] A. Schinzel, "On some problems of the arithmetical theory of continued fractions", *Acta Arith.* **VI** (1961), 393–413, and "On some problems of the arithmetical theory of continued fractions II", *Acta Arith.* **VII** (1962), 287–298.

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Exercise.

(a) Polynomials are usually thought of as having a basis consisting of the powers $1\,,\ X\,,\ X^2\,,\ X^3\,,\ \dots$ of the variable. So if $F(W)$ takes only integers values it seems natural to guess that all its coefficients must be integral. Not. Just as good a basis is given by the running powers $1\,,\ X\,,\ \frac{1}{2}X(X+1)\,,\ \frac{1}{6}X(X+1)(X+2)\,,\ \dots$ so a polynomial of degree s may have denominators as large as $s!$ in its usual presentation, yet take only integer values. Can one do better yet?

(b) Show that a polynomial F of even degree and with square leading coefficient may be written uniquely as $F = G^2 + 4R$, where the 'remainder' polynomial $4R$ has degree less than that of the polynomial G.

(c) Hence, this is not at all dead obvious, show if F is not the square of a polynomial, equivalently if R is not identically zero, that $F(H)$ cannot be a square for *any* sufficiently large integer H. It may here be useful to recognise that a polynomial of degree s evaluated at H has size of order H^s.

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$$\big(Y + (aW + b/2a)\big)\big(\overline{Y} + (aW + b/2a)\big) = (b^2 - 4a^2c)/4a^2$$

already displays a unit in $\mathcal{R} := \mathbb{Q}[W, Y]$; because any nonzero constant divides 1 in $\mathbb{Q}[W]$.

Exercise. (a) Verify that there is a unit in $\mathcal{R}$ of norm dividing 4 if and only if $b^2 - 4a^2c$ divides both $16a^4$ and $4b^2$. (b) Prove that $b = 0$ and $c | 4a$ in fact yields all the cases above (unless c and a^2 share an odd square factor). (i) Show that there is no loss of generality in presuming that both a and b are even. (ii) Hence replace $b \leftarrow 2b$, and note the the condition becomes $b^2 - a^2c$ divides both $4a^4$ and $4b^2$. (iii) Confirm there is now no loss of generality whatsoever in assuming that $0 \le b < |a|$. (iv) Show that each case with $b \ne 0$ corresponds to cases with shorter period than the case $b = 0$. (v) If $b = 0$ deduce there is a unit in $\mathcal{R}$ of norm dividing 4 for all integers W if and only if $c | 4a^2$. (vi) Show that if p is an odd prime, then p times a short period is always at least as long.

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Exercise. (a) Verify that there is a unit in $\mathcal{R}$ of norm dividing 4 if and only if $b^2 - 4a^2c$ divides both $16a^4$ and $4b^2$. (b) Prove that $b = 0$ and $c \mid 4a$ in fact yields all the cases above (unless c and a^2 share an odd square factor). (i) Show that there is no loss of generality in presuming that both a and b are even. (ii) Hence replace $b \leftarrow 2b$, and note the the condition becomes $b^2 - a^2c$ divides both $4a^4$ and $4b^2$. (iii) Confirm there is now no loss of generality whatsoever in assuming that $0 \le b < |a|$. (iv) Show that each case with $b \neq 0$ corresponds to cases with shorter period than the case $b = 0$. (v) If $b = 0$ deduce there is a unit in $\mathcal{R}$ of norm dividing 4 for all integers W if and only if $c \mid 4a^2$. (vi) Show that if p is an odd prime, then p times a short period is always at least as long.

Short Periods in Detail

I act on theory and experience by primarily considering ω given by

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I obtain the periods of $\sqrt{a^2W^2 + 4c}$ with $c \mid a$, accordingly.

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so, after a simple division by 2, if aW is odd

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Note that I do not observe the requirement that partial quotients should be positive, both so as to leave some exercises and to make clear that results that appear in the literature as dozens of distinct cases are in fact just a handful of cases.

Exercise.

(a) I speak of "dozens of different cases". If the computations above were completed by rewriting each expansion so that it has only positive partial quotients, and with c both positive and negative, how many different cases do in fact result?

(b) Rewrite several of the cases.

(c) Suppose $u = a + \omega b$ is a unit of $\mathbb{Z}[\omega]$ and set $u^h = a^{(h)} + \omega b^{(h)}$. If both $D = t^2 - 4n$ and $b = b^{(1)}$ are odd show that $b^{(k)}$ is even if and only if 3 divides k.

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