1. Let \( K = \mathbb{Q}(\sqrt{-d}), d > 1 \) squarefree.

(a) Prove that \( \mathcal{O}_K \) is not a UFD if one of the following holds:
   (i) \( d \equiv 1 \pmod{4}, d > 1 \);
   (ii) \( d \equiv 2 \pmod{4}, d > 2 \);
   (ii) \( d \equiv 7 \pmod{8}, d > 7 \).

(b) If \( \mathcal{O}_K \) is a UFD and \( d > 3 \) and \( d \equiv 3 \pmod{8} \), prove that \( d \) is a prime and that
    \[ x^2 + x + \frac{d+1}{4} \]
    assumes prime values for \( x = 0, \ldots, \frac{d-7}{4} \).

2. (Chinese Remainder Theorem) Let \( A \) and \( B \) be ideals of \( \mathcal{O}_K \) with \((A,B) = (1)\).
   Prove that the mapping \( f: \mathcal{O}_K/AB \to \mathcal{O}_K/A \oplus \mathcal{O}_K/B \), given by
   \[ f(x + AB) = (x + A, x + B) \]
   is well-defined and an isomorphism.

3. A commutative ring is called reduced if \( x^n = 0, n \in \mathbb{N} \Rightarrow x = 0 \). Prove that if
   \((A,B) = (1)\), that \( \mathcal{O}_K/AB \) is reduced if and only if \( \mathcal{O}_K/A \) and \( \mathcal{O}_K/B \) are reduced.
   If \( P \) is a prime ideal of \( \mathcal{O}_K \), prove that \( \mathcal{O}_K/P^e \) is reduced if and only if \( e = 1 \).
   Deduce that \( \mathcal{O}_K/(p) \) is not reduced if and only if \( p \) ramifies (ie. one of the prime
   ideal factors of \( (p) \) occurs to an exponent \( e > 1 \).)
   (P. Samuel uses this result in his book \textit{Algebraic theory of numbers} to prove Dedekind’s
   theorem: \( p \) ramifies if and only if \( p|D_K \).)

4. Find the group structure of the multiplicative group of equivalence classes of ideals
   in \( \mathbb{Q}(\sqrt{-21}) \).

5. Let \( K = \mathbb{Q}(\sqrt{34}) \).

   (a) Determine which primes must be examined in order to determine \( I_K \).

   (b) Use the Kummer–Dedekind theorem to factorize the principal ideals \( (2), (3) \)
   and \( (5) \):
   \[ (2) = P_2^2, \quad (3) = P_3Q_3, \quad (5) = P_5Q_5. \]

   (c) Use the equation \( N_K(6 + \sqrt{34}) = 2 \) to prove that \( P_2 = (6 + \sqrt{34}) \).

   (d) Let \( \alpha = 7 + \sqrt{34} \). With a suitable choice of labelling, prove that \( \alpha \in P_3 \) and
   \( \alpha \in P_5 \) and deduce that
   \[ P_3P_5 = (\alpha). \]

   (e) Prove that \( P_3^2 = (-5 + \sqrt{34}) \).

   (f) Given that \( \eta = 35 + 6\sqrt{34} \) is the fundamental unit of \( K \) and that \( N_K(\eta) = 1 \),
   prove that \( P_3 \) is not principal.

   (g) Determine the structure of the class group \( I_K \).
6. Suppose that $m \equiv 3 \pmod{8}$, $m$ is a prime and that $x^2 + x + \frac{m+1}{4}$ assumes prime values for $x = 0, 1, \ldots, \frac{m-3}{4}$. Prove that $\mathbb{Q}((\sqrt{-m}))$ is a UFD by showing that all ideals are principal.

7. Let $K = \mathbb{Q}(\sqrt{d})$, where $d$ is a squarefree integer, $d \neq 1$.
   
   (i) If $d < 0$ and $g \in \mathbb{N}$, where $g < |D_K|/4$, prove that there does not exist an $\alpha \in O_K$ satisfying $N_K(\alpha) = g$, unless $g = m^2$, $m \in \mathbb{N}$ and $\alpha = \pm m$.
   
   (ii) Let $d = -14$. Prove that if $J = (3, 1 + \sqrt{-14})$ and $K = (2, \sqrt{-14})$, then
   
   $J^2 = (9, -2 + \sqrt{-14})$, and $J^2 K = (-2 + \sqrt{-14})$.
   
   Also prove that $J^4$ is principal and $J^2$ is not principal. (Use part (i).)
   
   (iii) Prove that $I_K \cong C_4$.

8. In $\mathbb{Z}[\sqrt{-5}]$, let
   
   $A = (3, 4 + \sqrt{-5}), B = (3, 4 - \sqrt{-5}), C = (7, 4 + \sqrt{-5}), D = (7, 4 - \sqrt{-5})$.
   
   Show that $AB = (3), CD = (7), AC = (4 + \sqrt{-5}), BD = (4 - \sqrt{-5})$ and that $A, B, C$ and $D$ are prime ideals.
   
   Factorize $(1 + 2\sqrt{-5})$.

Please hand in Question 5 as Assignment 4.