PROBLEMS, SHEET 4, MP473, 2000

1. Let $K = \mathbb{Q}(\sqrt[4]{2})$. Find all isomorphisms $\sigma : K \to \mathbb{C}$ and the minimum polynomials and field polynomials of $\sqrt[4]{2}$, $\sqrt{2}$, 2, $\sqrt{2}+1$.

2. $(*)$ Let θ be an algebraic integer of degree n and let D be the discriminant of θ. Let F_k be the additive group

$$F_k = \mathbb{Z}\frac{1}{D} \oplus \mathbb{Z}\frac{\theta}{D} \oplus \cdots \oplus \mathbb{Z}\frac{\theta^k}{D},$$

$k = 0, \ldots, n-1$ and let $R_k = F_k \cap O_K$, where $K = \mathbb{Q}(\theta)$.

   (a) Prove that $R_0 = \mathbb{Z}$ and $R_{n-1} = O_K$.

   (b) Let d_k be the least positive integer such that $d_k R_k \subseteq \mathbb{Z}[\theta]$. Show that $xR_k \subseteq \mathbb{Z}[\theta]$ implies $d_k|x$.
   Show that $d_0 = 1$ and $d_k|d_{k+1}|D$.

   (c) Noting that the elements of R_k have the form

$$\alpha_k = \frac{a_{k0} + a_{k1}\theta + \cdots + a_{kk}\theta^k}{d_k},$$

   where $a_{k0}, \ldots, a_{kk} \in \mathbb{Z}$, define a *minimal integer of degree* k to be an element of R_k with least positive a_{kk}. Use induction on k to prove that $a_{kk} = 1$.

   (d) If $1, \alpha_1, \ldots, \alpha_{n-1}$ are minimal integers of degrees $0, 1, \ldots, n-1$ respectively, prove that they form an integral basis of K.

   (e) Prove that the index of θ is $d_1 \cdots d_{n-1}$. Also show that D is divisible by $d_k^{2(n-k)}$.

   (f) Show that the α_k can be chosen so that $0 \leq a_{kj} < d_k/d_j$ if $k > j$.

3. Use the above question to do Question 6, Sheet 3.

4. Let θ be a root of $x^3 - ax^2 - (a+3)x - 1$, where $m = a^2 + 3a + 9$ is squarefree.

   (a) Prove that $\Delta_K(1, \theta, \theta^2) = m^2$ and deduce that $1, \theta, \theta^2$ form an integral basis.

   (b) Prove that the conjugates of θ are $\theta' = -1/(1+\theta)$ and $\theta'' = -1/(1+\theta')$.

   (c) Show that θ and $1+\theta$ are units. (They are in fact a pair of fundamental units.)

5. Let $\zeta = e^{2\pi i/5}$, $K = \mathbb{Q}(\zeta)$ and $u = -\zeta^2(1+\zeta)$.

   (a) Show that $u \in U_K$.

   (b) Show that $1 < u < 2$.

   (c) Show that $\mathbb{R} \cap \mathbb{Q}(\zeta) = \mathbb{Q}(\sqrt{5})$.

   (d) Use (a),(b),(c) to prove that $u = (1+\sqrt{5})/2$.

(e) Prove that u is a fundamental unit of K. List the units of K.

6. If $d > 0$ is squarefree and $d \equiv 1 \pmod{8}$, prove that there is no unit of $\mathbb{Q}(\sqrt{d})$ of the form $(x + y\sqrt{d})/2$, where x and y are odd.

7. Find the fundamental unit of $\mathbb{Q}(\sqrt{69})$. (Ans: $11 + 3\omega$.)

8. If $\alpha \in K$ is a root of a monic polynomial $f(x) \in \mathbb{Z}[x]$ and if $f(r) = \pm 1$, where $r \in \mathbb{Z}$, show that $\alpha - r \in U_K$.