1. Let $K = \mathbb{Q}(\sqrt{2})$. Find all isomorphisms $\sigma : K \to \mathbb{C}$ and the minimum polynomials and field polynomials of $\sqrt{2}$, $\sqrt{2} + 2$, $\sqrt{2} + 1$.

2. (*) Let $\theta$ be an algebraic integer of degree $n$ and let $D$ be the discriminant of $\theta$. Let $F_k$ be the additive group

$$F_k = \mathbb{Z} \frac{1}{D} \oplus \mathbb{Z} \frac{\theta}{D} \oplus \cdots \oplus \mathbb{Z} \frac{\theta^k}{D},$$

$k = 0, \ldots, n-1$ and let $R_k = F_k \cap O_K$, where $K = \mathbb{Q}(\theta)$.

(a) Prove that $R_0 = \mathbb{Z}$ and $R_{n-1} = O_K$.

(b) Let $d_k$ be the least positive integer such that $d_k R_k \subseteq \mathbb{Z}[\theta]$. Show that $xR_k \subseteq \mathbb{Z}[\theta]$ implies $d_k | x$.

(c) Noting that the elements of $R_k$ have the form

$$a_k = \frac{a_{k0} + a_{k1} \theta + \cdots + a_{kk} \theta^k}{d_k},$$

where $a_{k0}, \ldots, a_{kk} \in \mathbb{Z}$, define a minimal integer of degree $k$ to be an element of $R_k$ with least positive $a_{kk}$. Use induction on $k$ to prove that $a_{kk} = 1$.

(d) If $1, \alpha_1, \ldots, \alpha_{n-1}$ are minimal integers of degrees $0, 1, \ldots, n-1$ respectively, prove that they form an integral basis of $K$.

(e) Prove that the index of $\theta$ is $d_1 \cdots d_{n-1}$. Also show that $D$ is divisible by $d_k^{2(n-k)}$.

(f) Show that the $\alpha_k$ can be chosen so that $0 \leq a_{kj} < d_k/d_j$ if $k > j$.

3. Use the above question to do Question 6, Sheet 3.

4. Let $\theta$ be a root of $x^3 - ax^2 - (a + 3)x - 1$, where $m = a^2 + 3a + 9$ is squarefree.

(a) Prove that $\Delta_K(1, \theta, \theta^2) = m^2$ and deduce that $1, \theta, \theta^2$ form an integral basis.

(b) Prove that the conjugates of $\theta$ are $\theta' = -1/(1 + \theta)$ and $\theta'' = -1/(1 + \theta')$.

(c) Show that $\theta$ and $1 + \theta$ are units. (They are in fact a pair of fundamental units.)

5. Let $\zeta = e^{2\pi i/5}$, $K = \mathbb{Q}(\zeta)$ and $u = -\zeta^2(1 + \zeta)$.

(a) Show that $u \in U_K$.

(b) Show that $1 < u < 2$.

(c) Show that $\mathbb{R} \cap \mathbb{Q}(\zeta) = \mathbb{Q}(\sqrt{5})$.

(d) Use (a),(b),(c) to prove that $u = (1 + \sqrt{5})/2$. 

1
(e) Prove that $u$ is a fundamental unit of $K$. List the units of $K$.

6. If $d > 0$ is squarefree and $d \equiv 1 \pmod{8}$, prove that there is no unit of $\mathbb{Q}(\sqrt{d})$ of the form $(x + y\sqrt{d})/2$, where $x$ and $y$ are odd.

7. Find the fundamental unit of $\mathbb{Q}(\sqrt{69})$. (Ans: $11 + 3\omega$.)

8. If $\alpha \in K$ is a root of a monic polynomial $f(x) \in \mathbb{Z}[x]$ and if $f(r) = \pm 1$, where $r \in \mathbb{Z}$, show that $\alpha - r \in U_K$. 