Problem Sheet 3, MP473, Semester 2, 2000

1. Let θ be a root of the irreducible polynomial $x^3 + 11x + 4$.

   Verify that $\Delta_K(1, \theta, \theta^2) = -4 \cdot 1439$ and prove that
$$1,\, \theta,\, \frac{(\theta + \theta^2)}{2}$$
   form an integral basis for $K = \mathbb{Q}(\theta)$. Also find D_K.

2. Let $\theta \in \mathbb{C}$ satisfy $\theta^3 + 2\theta - 6 = 0$.
   (i) Calculate $N_K(3\theta + 2)$ and $\Delta_K(1, \theta, \theta^2)$.
   (ii) Use the Eisenstein lemma to get the exact value numerical value of D_K. (Warning: The Stickelberger criterion does not apply.)

3. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
   (i) Let $\theta = (\sqrt{2} + \sqrt{6})/2$. Verify that $\Delta_K(1, \theta, \theta^2, \theta^3) = 2^8 \cdot 3^2$.
   (ii) Let $\phi = \theta + 1$. Compute $m_\phi(x)$ and hence prove $D_K = 2^8 \cdot 3^2$. (Hint: $L = \mathbb{Q}(\sqrt{3}) \subseteq K$. Then use the fact that $D_L^{[K:L]}$ divides D_K.)
   (iii) Verify that $\Delta_K(1, \sqrt{2}, \sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2}) = 2^8 \cdot 3^2$.

4. With respect to the field L of Question 7, Sheet 2:
   (i) Use the fact that $\zeta, \ldots, \zeta^{p-1}$ form an integral basis for K to prove that $1, \omega_1, \ldots, \omega_{(p-3)/2}$ and $1, \omega, \ldots, \omega^{(p-3)/2}$ form integral bases for L.
   (ii) Prove that $T_L(\omega_r) = -1$ if $p \not| r$.
   (iii) Prove that
$$T_L(\omega_r\omega_s) = \begin{cases} p-2 & \text{if } 1 \le r = s \le (p-3)/2, \\ -2 & \text{if } 1 \le r \ne s \le (p-3)/2 \end{cases}$$
   and deduce that $D_L = p^{(p-3)/2}$.

5. Let $\omega_1, \ldots, \omega_n$ belong to O_K and form a $\mathbb{Q}$–basis for K. Show that every $\theta \in O_K$ can be written as
$$\theta = \sum_{j=1}^{n} \frac{x_j}{\Delta_K(\omega_1, \ldots, \omega_n)} \omega_j,$$
   where x_j and $x_j^2/\Delta_K(\omega_1, \ldots, \omega_n)$ are rational integers. (Hint: Use Cramer's rule.)

6. (hard) With respect to problem 3, Sheet 2, prove that $1,\ \theta,\ \theta^2$ form an integral basis for K. (Hint: use a remark made at the end of the proof of the Eisensteinian proof, together with the previous question.)

Problems 1,2,5 to be handed in as ASSIGNMENT 2.

(Handed out Friday 25th August 2000)