Problem Sheet 2, MP473, Semester 2, 2000

1. Let $\omega = \zeta + \zeta^{-1} = 2\cos\frac{2\pi}{5}$, where $\zeta = e^{\frac{2\pi i}{5}}$.

   Prove that $w = (-1+\sqrt{5})/2$.

2. Algebraic numbers α and β have degrees m and n, where $\gcd(m,n) = 1$. Prove that $[\mathbb{Q}(\alpha,\beta):\mathbb{Q}] = mn$.

   Let K be the splitting field of $(x^2+1)(x^3-2)$. Find $[K:\mathbb{Q}]$.

3. Let $f(x) = x^3 - 3x + 1$ and θ a root of $f(x)$. Let $K = \mathbb{Q}(\theta)$.

   (a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.

   (b) Show that $\theta^2 - 2$ is also a root of $f(x)$ and find the other root.

   (c) Calculate $N_K(\theta), T_K(\theta), N_K(\theta^2), T_K(\theta^2)$.

   (d) Prove that K is a normal extension of $\mathbb{Q}$.

   (e) List the elements of $\mathrm{Gal}(K/\mathbb{Q})$ and identify the group.

4. Calculate $\Phi_{200}(x)$.

5. G is a *discrete* additive group of real numbers if for each $n \in \mathbb{N}$, there are only finitely many group elements g satisfying $|g| \le n$. Prove that G is an infinite cyclic group generated by the least positive element in G.

6. Let $\zeta_n = e^{2\pi i/n}$. If $\gcd(m,n) = 1$, prove that

$$\mathbb{Q}(\zeta_{mn}) = \mathbb{Q}(\zeta_m, \zeta_n)$$

7. Let $\zeta = e^{2\pi i/p}$, $p > 3$ a prime and let

$$\omega_r = \zeta^r + \zeta^{-r}, \quad r \ge 1,\ p \nmid r.$$

   Also let $\omega = \omega_1$, $K = \mathbb{Q}(\zeta)$ and $L = \mathbb{Q}(\omega)$.

   (i) Prove that $K \cap \mathbb{R} = L$. [Hint: Use the fact that $\zeta, \ldots, \zeta^{(p-1)}$ form a $\mathbb{Q}$–basis for K.]

   (ii) Prove that $x^2 - \omega x + 1$ is irreducible over L and that its roots are ζ and ζ^{-1}. Deduce that $[L:\mathbb{Q}] = (p-1)/2$.

   (iii) Use CMAT to find $m_\omega(x)$ when $p = 5, 7, 11$, by factorizing the field polynomial of ω in K over $\mathbb{Q}[x]$.

   (iv) Let σ_r be the $\mathbb{Q}$–automorphism of K defined by $\sigma_r(\zeta) = \zeta^r$, $p \nmid r$. Prove that $\sigma_r(\omega) \ne \sigma_s(\omega)$ if $1 \le r < s \le (p-1)/2$, but that $\sigma_r(\omega) = \sigma_{p-r}(\omega)$ if $1 \le r \le (p-1)/2$. Deduce that L is a normal extension of $\mathbb{Q}$.

Problems 1,2,3,4 to be handed in as ASSIGNMENT 1.

(Handed out Friday 18th August 2000)