## Problems, Sheet 1, MP473, Semester 2, 2000

1. Let $m$ be a positive integer, not a perfect square. Prove that the Jacobi symbol $\left(\frac{a}{m}\right)$ has the property that there is an integer $b$ such that $\left(\frac{b}{m}\right) = -1$ and deduce that
$$S = \sum_{a=1}^{m-1} \left(\frac{a}{m}\right) = 0.$$
[Hint: Consider $\left(\frac{b}{m}\right) S$.]

2. Let $m$ and $k$ be integers, $k$ positive.
$$g(m,k) = \sum_{t=0}^{k-1} e^{\frac{2\pi i t^2 m}{k}}.$$
If $\gcd(k_1, k_2) = 1$ and $k_1 \geq 1, k_2 \geq 1$, $m \in \mathbb{Z}$, prove that
$$g(mk_1, k_2) g(mk_2, k_1) = g(m, k_1 k_2).$$
State a generalisation to $g(m, k_1 k_2 \cdots k_n)$, where $\gcd(k_i, k_j) = 1$ if $i \neq j$.

3. Let $p$ be an odd prime not dividing $m$. Prove that if $b \geq 2$,
$$g(m, p^b) = p g(m, p^{b-2}) = \begin{cases} p^{\frac{b}{2}} & \text{if } b \text{ is even,} \\ p^{\frac{b-1}{2}} g(m,p) & \text{if } b \text{ is odd.} \end{cases}$$
[Hint: If $0 \leq k < p^b$, write $k = p^{b-1}z + t$, $0 < z < p$, $0 < t < p^{b-1}$.]

4. Use the previous exercises, together with the case $k$ prime (proved in lectures) to prove that if $\gcd(m,k) = 1$, $m, k$ integers, $k$ odd, $k > 1$, then
$$g(m,k) = \begin{cases} \left(\frac{m}{k}\right) \sqrt{k} & \text{if } k \equiv 1 \pmod{4}, \\ \left(\frac{m}{k}\right) i\sqrt{k} & \text{if } k \equiv -1 \pmod{4}. \end{cases}$$

5. Let $m$ be odd. Prove that

   (i) $g(m, 4) = 2(1 + i^m)$;

   (ii) $g(m, 8) = 4e^{\frac{\pi i m}{4}} = \left(\frac{2}{m}\right)(1 + i^m) 2^{\frac{3}{2}}$,

   where $\left(\frac{2}{m}\right) = \left(\frac{2}{|m|}\right)$.

6. If $s \geq 4$ and $b = 2^s$, prove that
$$g(m,b) = 2g(m, b/4) = \begin{cases} 2^{\frac{s-2}{2}} g(m,4) & \text{if } s \text{ is even,} \\ 2^{\frac{s-3}{2}} g(m,8) & \text{if } s \text{ is odd} \end{cases}.$$

by writing

$$g(m,b) = \sum_{\substack{k=0 \\ k \text{ odd}}}^{b-1} e^{\frac{2\pi i k^2 m}{b}} + \sum_{\substack{k=0 \\ k \text{ even}}}^{b-1} e^{\frac{2\pi i k^2 m}{b}}.$$

Show that the first sum is 0 by noting that

$$e^{\frac{2\pi i (k+\frac{b}{4})^2 m}{b}} = -e^{\frac{2\pi i k^2 m}{b}}.$$

Also deduce that if $s \geq 2$, then

$$g(m, 2^s) = \left(\frac{2^s}{m}\right)(1 + i^m)2^{\frac{s}{2}}.$$

7. Let $\gcd(m,b) = 1$, $m$ and $b$ integers, $b > 0$ and $4|b$. Prove that

$$g(m,b) = \left(\frac{b}{m}\right)(1 + i^m)\sqrt{b}.$$