1. Let $m$ be a positive integer, not a perfect square. Prove that the Jacobi symbol $(\frac{a}{m})$ has the property that there is an integer $b$ such that $(\frac{b}{m}) = -1$ and deduce that

$$S = \sum_{a=1}^{m-1} \left( \frac{a}{m} \right) = 0.$$  

[Hint: Consider $(\frac{b}{m}) S$.]

2. Let $m$ and $k$ be integers, $k$ positive.

$$g(m, k) = \sum_{t=0}^{k-1} e^{\frac{2\pi it^2}{k}}.$$  

If $\gcd(k_1, k_2) = 1$ and $k_1 \geq 1, k_2 \geq 1, m \in \mathbb{Z}$, prove that

$$g(mk_1, k_2)g(mk_2, k_1) = g(m, k_1k_2).$$  

State a generalisation to $g(m, k_1k_2 \cdots k_n)$, where $\gcd(k_i, k_j) = 1$ if $i \neq j$.

3. Let $p$ be an odd prime not dividing $m$. Prove that if $b \geq 2$,

$$g(m, p^b) = pg(m, p^{b-2}) = \begin{cases} p^{\frac{b}{2}} & \text{if } b \text{ is even}, \\ p^{\frac{b-1}{2}}g(m, p) & \text{if } b \text{ is odd}. \end{cases}$$  

[Hint: If $0 \leq k < p^b$, write $k = p^{b-1}z + t, \ 0 < z < p, \ 0 < t < p^{b-1}$.]

4. Use the previous exercises, together with the case $k$ prime (proved in lectures) to prove that if $\gcd(m, k) = 1, m, k \text{ integers}, k \text{ odd}, k > 1$, then

$$g(m, k) = \begin{cases} \left( \frac{m}{k} \right) \sqrt{k} & \text{if } k \equiv 1 \pmod{4}, \\ \left( \frac{m}{k} \right) i\sqrt{k} & \text{if } k \equiv -1 \pmod{4}. \end{cases}$$

5. Let $m$ be odd. Prove that

(i) $g(m, 4) = 2(1 + im)$;

(ii) $g(m, 8) = 4e^{\frac{\pi im}{4}} = \left( \frac{2}{m} \right) (1 + im)2^{\frac{3}{2}},$

where $\left( \frac{2}{m} \right) = \left( \frac{2}{|m|} \right)$.

6. If $s \geq 4$ and $b = 2^s$, prove that

$$g(m, b) = 2g(m, b/4) = \begin{cases} 2^{\frac{s-2}{2}}g(m, 4) & \text{if } s \text{ is even}, \\ 2^{\frac{s-3}{2}}g(m, 8) & \text{if } s \text{ is odd}. \end{cases}$$
by writing
\[
g(m, b) = \sum_{\substack{k = 0 \atop k \text{ odd}}}^{b-1} e^{\frac{2\pi ik^2m}{b}} + \sum_{\substack{k = 0 \atop k \text{ even}}}^{b-1} e^{\frac{2\pi ik^2m}{b}}.
\]

Show that the first sum is 0 by noting that
\[
e^{\frac{2\pi i(k + \frac{1}{2})^2m}{b}} = -e^{\frac{2\pi ik^2m}{b}}.
\]

Also deduce that if \( s \geq 2 \), then
\[
g(m, 2^s) = \left(\frac{2^s}{m}\right) (1 + i^m)2^s.
\]

7. Let \( \gcd(m, b) = 1 \), \( m \) and \( b \) integers, \( b > 0 \) and \( 4 \mid b \). Prove that
\[
g(m, b) = \left(\frac{b}{m}\right) (1 + i^m)\sqrt{b}.
\]