1. (a) Explain what is meant by the statements: (i) \( \theta \) is an algebraic integer, (ii) \( K \) is an algebraic number field of degree \( n \), (iii) \( \omega_1, \ldots, \omega_n \) is an integral basis for \( K \), \( u \) is a unit of the ring \( O_K \), (iv) \( D_K \), the discriminant of \( K \), (v) \( T_K(\alpha) \), (vi) \( N_K(\alpha) \).

(b) Write down integral bases and discriminants for \( \mathbb{Q}(\sqrt{d}) \), where \( d \) is squarefree and for \( \mathbb{Q}(\sqrt[p]{\eta}) \), where \( p \) is an odd prime.

(c) Let \( \omega = \zeta + \zeta^{-1} = 2 \cos \frac{2\pi}{5}, \) where \( \zeta = e^{\frac{2\pi i}{5}} \).
Prove that \( u = (-1 + \sqrt{5})/2 \).

(d) Let \( \alpha \in \mathbb{Q}(\sqrt{d}), d \) squarefree, \( d \neq 1 \). If \( N_K(\alpha) \) and \( T_K(\alpha) \) belong to \( \mathbb{Z} \), prove that \( \alpha \) is an algebraic integer.

(e) Prove that if \( u \in O_K \), then \( u \) is a unit of \( O_K \) if and only if \( N_K(u) = \pm 1 \). Find the smallest unit \( \eta > 1 \) in \( O_K \) when \( K = \mathbb{Q}(\sqrt{21}) \).

2. (a) State Stickelberger’s theorem.

(b) If \( K \) is a number field of degree \( n \) and \( \omega_1, \ldots, \omega_n \) are \( \mathbb{Q} \)-linearly independent, define \( \Delta_K(\omega_1, \ldots, \omega_n) \).

(c) Let \( \theta = \sqrt[3]{2} \).

(i) Prove that \( \Delta_K(1, \theta, \theta^2) = -27 \times 4 \). Deduce that \( D_K = -27 \times 4 \) or \(-3 \times 4 \), using the fact that \( m_\theta(x) \) is Eisensteinian with respect to \( p = 2 \).

(ii) Let \( \phi = \theta - 2 \). Find \( m_\phi(x) \), observe that it is Eisensteinian with respect to \( p = 3 \) and finally deduce that \( D_K = -27 \times 4 \).

3. (a) Define the terms ideal, prime ideal in \( O_K \).

(b) If \( A \) and \( B \) are ideals in \( O_K \), prove that \( A + B \) is an ideal of \( O_K \).

(c) Define \( N(A) \), the norm of the ideal \( A \) and prove that if \( N(A) = p \), where \( p \) is a prime, then \( A \) is a prime ideal.

(d) If \( K = \mathbb{Q}(\sqrt{-5}) \), prove that \( (8 + 3\sqrt{-5}, 3 + 4\sqrt{-5}) = (1) \).

(e) Let \( d \) be a squarefree integer, \( d \equiv 2 \mod 4 \), \( K = \mathbb{Q}(\sqrt{d}) \). Also let \( P = (2, \sqrt{d}) \). Without using the Kummer–Dedekind theorem,

(i) Prove that \( P^2 = (2) \).

(ii) Let \( \alpha = a + b\sqrt{d} \in O_K \). Prove that \( \alpha \in P \) if and only if \( 2 | a \).
Deduce that \( P \) is a prime ideal in \( O_K \).

4. (a) Define \( I_K \), the ideal class group of \( K \) and give a method for determining a family of generators for \( I_K \).

(b) Let \( K = \mathbb{Q}(\sqrt{-23}) \). Also let \( \omega = \frac{1 + \sqrt{-23}}{2} \) and \( \omega' = \frac{1 - \sqrt{-23}}{2} \). Let \( P_2 = (2, \omega), Q_2 = (2, \omega'), P_3 = (3, \omega), Q_3 = (3, \omega') \).

(i) Prove that \( P_2Q_2 = (2) \), \( P_3Q_3 = (3) \) and that \( P_2, Q_2, P_3, Q_3 \) are prime ideals of norms \( 2, 2, 3, 3 \), respectively.
(ii) Prove that $2 - \omega \in P_2$ but $2 - \omega \not\in Q_2$ and using this fact, or otherwise, noting the equation

$$8 = (2 - \omega)(2 - \omega'),$$

deduce that $P_2^3 = (2 - \omega)$.

(iii) Prove that $P_2P_3 = (\omega)$.

(iv) Determine the structure of $I_K$.

5. Prove that $\gcd(\sqrt{2}x - 1, \sqrt{2}x + 1) = 1$ in the UFD $\mathbb{Z}[\sqrt{2}]$.  