MP473 Examination, November 1998

Time: 3 hours

Answer all questions

1. (a) Explain what is meant by the statements: (i) θ *is an algebraic integer*, (ii) K *is an algebraic number field of degree* n, (iii) $\omega_1, \ldots, \omega_n$ *is an integral basis for* K, u *is a unit of the ring* O_K, (iv) D_K, *the discriminant of* K, (v) $T_K(\alpha)$, (vi) $N_K(\alpha)$.

   (b) Write down integral bases and discriminants for $\mathbb{Q}(\sqrt{d})$, where d is squarefree and for $\mathbb{Q}(e^{\frac{2\pi i}{p}})$, where p is an odd prime.

   (c) Let $\omega = \zeta + \zeta^{-1} = 2\cos\frac{2\pi}{5}$, where $\zeta = e^{\frac{2\pi i}{5}}$.
   Prove that $w = (-1+\sqrt{5})/2$.

   (d) Let $\alpha \in \mathbb{Q}(\sqrt{d})$, d squarefree, $d \neq 1$. If $N_K(\alpha)$ and $T_K(\alpha)$ belong to $\mathbb{Z}$, prove that α is an algebraic integer.

   (e) Prove that if $u \in O_K$, then u is a unit of O_K if and only if $N_K(u) = \pm 1$. Find the smallest unit $\eta > 1$ in O_K when $K = \mathbb{Q}(\sqrt{21})$.

2. (a) State Stickelberger's theorem.

   (b) If K is a number field of degree n and $\omega_1, \ldots, \omega_n$ are $\mathbb{Q}$–linearly independent, define $\Delta_K(\omega_1, \ldots, \omega_n)$.

   (c) Let $\theta = \sqrt[3]{2}$.

      (i) Prove that $\Delta_K(1, \theta, \theta^2) = -27 \times 4$. Deduce that $D_K = -27 \times 4$ or -3×4, using the fact that $m_\theta(x)$ is Eisensteinian with respect to $p = 2$.

      (ii) Let $\phi = \theta - 2$. Find $m_\phi(x)$, observe that it is Eisensteinian with respect to $p = 3$ and finally deduce that $D_K = -27 \times 4$.

3. (a) Define the terms *ideal, prime ideal* in O_K.

   (b) If A and B are ideals in O_K, prove that $A + B$ is an ideal of O_K.

   (c) Define $N(A)$, the *norm* of the ideal A and prove that if $N(A) = p$, where p is a prime, then A is a prime ideal.

   (d) If $K = \mathbb{Q}(\sqrt{-5})$, prove that $(8 + 3\sqrt{-5}, 3 + 4\sqrt{-5}) = (1)$.

   (e) Let d be a squarefree integer, $d \equiv 2 \pmod 4$, $K = \mathbb{Q}(\sqrt{d})$. Also let $P = (2, \sqrt{d})$. Without using the Kummer–Dedekind theorem,

      (i) Prove that $P^2 = (2)$.

      (ii) Let $\alpha = a + b\sqrt{d} \in O_K$. Prove that $\alpha \in P$ if and only if $2|a$. Deduce that P is a prime ideal in O_K.

4. (a) Define I_K, the *ideal class group* of K and give a method for determining a family of generators for I_K.

   (b) Let $K = \mathbb{Q}(\sqrt{-23})$. Also let $\omega = \frac{1+\sqrt{-23}}{2}$ and $\omega' = \frac{1-\sqrt{-23}}{2}$. Let

   $$P_2 = (2, \omega), Q_2 = (2, \omega'), P_3 = (3, \omega), Q_3 = (3, \omega').$$

      (i) Prove that $P_2 Q_2 = (2)$, $P_3 Q_3 = (3)$ and that P_2, Q_2, P_3, Q_3 are prime ideals of norms $2, 2, 3, 3$, respectively.

(ii) Prove that $2 - \omega \in P_2$ but $2 - \omega \notin Q_2$ and using this fact, or otherwise, noting the equation

$$8 = (2 - \omega)(2 - \omega'),$$

deduce that $P_2^3 = (2 - \omega)$.

(iii) Prove that $P_2 P_3 = (\omega)$.

(iv) Determine the structure of I_K.

5. Prove that $\gcd(\sqrt{2}x - 1, \sqrt{2}x + 1) = 1$ in the UFD $\mathbb{Z}[\sqrt{2}]$.