MP473 EXAM, Semester 2, 1996

Attempt all questions

1. (In what follows, O_K denotes the ring of integers in an algebraic number field K, $[K : \mathbb{Q}] = n$ and $\theta \in K$.

   (a) Define the terms (i) $m_\theta(x)$, (ii) *integral basis of* K, (iii) $\Delta(1, \theta, \dots, \theta^{n-1})$, (iv) D_K and (v) $N_K(\theta)$.

   (b) Let $\theta \in \mathbb{C}$ satisfy $\theta^3 + 2\theta - 6 = 0$.

   (i) Calculate $N_K(\theta)$ and $N_K(3\theta + 2)$.

   (ii) Calculate $\Delta(1, \theta, \theta^2)$.

   (iii) Use the Eisenstein lemma to get the exact value numerical value of D_K. (Warning: The Stickelberger criterion does not apply.)

2. Let $\zeta = e^{2\pi i/p}$, $p > 3$ a prime and let

$$\omega_r = \zeta^r + \zeta^{-r}, \quad r \geq 1,\ p \not| r.$$

   Also let $\omega = \omega_1$, $K = \mathbb{Q}(\zeta)$ and $L = \mathbb{Q}(\omega)$.

   (a) Prove that $K \cap \mathbb{R} = L$. [Hint: Use the fact that $\zeta, \dots, \zeta^{p-1}$ form a $\mathbb{Q}$–basis for K.]

   (b) Prove that $x^2 - \omega x + 1$ is irreducible over L. Deduce that $[L : \mathbb{Q}] = (p-1)/2$.

   (c) Using CMAT, or otherwise, find $m_\omega(x)$ when $p = 5$.

   (d) Describe the $\mathbb{Q}$–isomorphisms of L and prove that L is a normal extension of $\mathbb{Q}$.

3. (a) If I is an ideal of O_K, define $N(I)$, the *norm* of I.

   (b) If $N(I) = p$, where p is a prime number, prove that I is a prime ideal. Give a counterexample to the converse statement.

   (c) Prove that $N(I)$ is always a prime power if I is prime ideal.

   (d) In $\mathbb{Q}(\sqrt{-5})$, if $I = (3, 4 + \sqrt{-5})$, prove from first principles that $N(I)$=3.

   (e) Let $K = \mathbb{Q}(\theta)$, where $\theta = \sqrt{10}$. Prove from first principles that if $I = (2, \theta)$, then $I^2 = (2)$. Also prove that I is not principal.

   (f) If $d \not\equiv 1 \pmod 4$ is squarefree, prove from first principles that if the Legendre symbol $\left(\frac{d}{p}\right) = -1$, then (p) is a prime ideal in O_K, $K = \mathbb{Q}(\sqrt{d})$.

4. (a) Define the ideal class group I_K of an algebraic number field K.

   (b) Use the fact that if B is a nonzero ideal of O_K, there exists an $\alpha \in B$, such that $|N_K(\alpha)| \leq (\frac{4}{\pi})^s \frac{n!}{n^n}(\sqrt{|D_K|})N(B)$, to prove that I_K is finite. (Hint: If A is an ideal of O_K, let B be an ideal satisfying $AB = (d), d \in \mathbb{N}$.)

(c) State the Kummer–Dedekind theorem describing the prime ideal decomposition of (p), where p is a prime number.

(d) Let $K = \mathbb{Q}(\sqrt{-17})$.

(i) Let $J = \overline{(3, 1+\sqrt{-17})}$. Prove that

$$J^2 = \overline{(2, 1+\sqrt{-17})},\ J^3 = \overline{(3, 1-\sqrt{-17})}.$$

(ii) Prove that the ideal class group of K is C_4, with generator J.

5. (a) If O_K is a UFD and α, β, γ are non–zero relatively prime elements of O_K satisfying $\alpha\beta = \gamma^3$, what can be said of α and β?

(b) Given that if $K = \mathbb{Q}(\sqrt{-2})$, then O_K is a principal ideal domain, describe the irreducible elements of O_K. Factorize $2, 3, 11$ and $5 + 2\sqrt{-2}$ in O_K.

(c) Let x and y be rational integers satisfying

$$x^2 + 2 = y^3. \qquad (1)$$

(i) If $K = \mathbb{Q}(\sqrt{-2})$, prove that $\gcd(x+\sqrt{-2}, x-\sqrt{-2}) = 1$ in the PID O_K.

(ii) By rewriting equation (1) as

$$(x+\sqrt{-2})(x-\sqrt{-2}) = y^3,$$

use (a) to deduce that $(x, y) = (\pm 5, 3)$.