1. Define the terms integral basis of $K$, $D_K$, the discriminant of $K$ and $\Delta(1, \theta, \ldots, \theta^{n-1})$, the discriminant of $\theta$, where $\theta \in K$.

Let $\theta = (\sqrt{2} + \sqrt{6})/2$ and $K = \mathbb{Q}(\theta)$.

(a) Prove that $\theta$ is an algebraic integer and find $m_\theta(x)$.
(b) Find $N_K(\theta), T_K(\theta), N_K(\theta^2 - 2)$ and also compute $\Delta(1, \theta, \theta^2, \theta^3)$.
(c) Let $\phi = \theta + 1$. Find $m_\phi(x)$. What does this tell us about $D_K$?
(d) Prove that $K$ is a normal extension of $\mathbb{Q}$.

2. (a) If $I$ is an ideal of $O_K$, define $N(I)$, the norm of $I$.
(b) If $N(I) = p$, where $p$ is a prime number, prove that $I$ is a prime.
(c) Prove that $N(I)$ is always a prime power if $I$ is prime ideal.
(d) Give an example of a quadratic field $K$ and a prime ideal $I$ of $O_K$, for which $N(I)$ is not a prime number.
(e) Let $I$ be an ideal of $O_K$ and let $\alpha \in I$ satisfy

$$N(I) = |N_K(\alpha)|.$$ 

Prove that $I = (\alpha)$.
(f) Let $K$ be a real quadratic field and suppose that the fundamental unit $\eta$ of $K$ satisfies $N_K(\eta) = 1$. Let $I$ be an ideal of $O_K$ with the property that

$$I^2 = (\alpha),$$ 

where $N_K(\alpha) < 0$. Prove that $I$ is not a principal ideal.

3. Define $I_K$, the ideal class group of $K$.

Let $K = \mathbb{Q}(\sqrt{34})$.

(a) Determine which primes must be examined in order to determine $I_K$.
(b) Use the Kummer–Dedekind theorem to factorize the principal ideals $(2), (3)$ and $(5)$:

$$(2) = P_2^2, \quad (3) = P_3Q_3, \quad (5) = P_5Q_5.$$ 

(c) Use the equation $N_K(6 + \sqrt{34}) = 2$ to prove that $P_2 = (6 + \sqrt{34})$.
(d) Let $\alpha = 7 + \sqrt{34}$. With a suitable choice of labelling, prove that $\alpha \in P_3$ and $\alpha \in P_5$ and deduce that

$$P_3P_5 = (\alpha).$$ 

(e) Prove that $P_3^2 = (-5 + \sqrt{34})$. 

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(f) Given that \( \eta = 35 + 6\sqrt{34} \) is the fundamental unit of \( K \) and that \( N_K(\eta) = 1 \), use Question 2(f) to prove that \( P_3 \) is not principal.

(g) Determine the structure of the class group \( I_K \).

4. (a) If \( a \in \mathbb{N} \), show that the polynomial \( x^3 + ax - 1 \) is irreducible over \( \mathbb{Q} \) and has only one real root \( \alpha \).

(b) Prove that \( \Delta(1, \alpha, \alpha^2) = -(4a^3 + 27) \).

(c) Suppose that \( 4a^3 + 27 \) is squarefree. Explain why \( D_K = -(4a^3 + 27) \) and prove that the fundamental unit \( \eta \) satisfies \( \eta > a \).

(d) Observe that \( \alpha \) is a unit between 0 and 1. Prove that \( a < \alpha^{-1} < a + 1 \) and deduce that \( \eta = \alpha^{-1} \) if \( a \geq 2 \).

5. (a) Let \( K = \mathbb{Q}(\sqrt{-5}) \). Prove that the ideal \( (3, 1 + \sqrt{-5}) \) is not principal.

(b) Let \( K = \mathbb{Q}(i) \).

(1) List the units of \( O_K \).

(2) If \( y = 2n + 1, n \in \mathbb{Z} \), prove that \( 1 + i \) divides \( y \pm i \). Assuming that \( \mathbb{Z}[i] \) is a UFD, prove that

\[
\gcd \left( \frac{y + i}{1 + i}, \frac{y - i}{1 + i} \right) = 1.
\]