MP473 EXAM, Semester 2, 1994

(In what follows, O_K denotes the ring of integers in an algebraic number field K. Also if $\theta \in K$, $m_\theta(x)$ denotes the minimum polynomial of θ.)

1. Define the terms *integral basis of* K, D_K, the *discriminant* of K and $\Delta(1, \theta, \ldots, \theta^{n-1})$, the *discriminant* of θ, where $\theta \in K$.

   Let $\theta = (\sqrt{2} + \sqrt{6})/2$ and $K = \mathbb{Q}(\theta)$.

   (a) Prove that θ is an algebraic integer and find $m_\theta(x)$.

   (b) Find $N_K(\theta), T_K(\theta), N_K(\theta^2 - 2)$ and also compute $\Delta(1, \theta, \theta^2, \theta^3)$.

   (c) Let $\phi = \theta + 1$. Find $m_\phi(x)$. What does this tell us about D_K?

   (d) Prove that K is a normal extension of $\mathbb{Q}$.

2. (a) If I is an ideal of O_K, define $N(I)$, the *norm* of I.

   (b) If $N(I) = p$, where p is a prime number, prove that I is a prime

   (c) Prove that $N(I)$ is always a prime power if I is prime ideal.

   (d) Give an example of a quadratic field K and a prime ideal I of O_K, for which $N(I)$ is not a prime number.

   (e) Let I be an ideal of O_K and let $\alpha \in I$ satisfy

   $$N(I) = |N_K(\alpha)|.$$

   Prove that $I = (\alpha)$.

   (f) Let K be a real quadratic field and suppose that the fundamental unit η of K satisfies $N_K(\eta) = 1$. Let I be an ideal of O_K with the property that

   $$I^2 = (\alpha),$$

   where $N_K(\alpha) < 0$. Prove that I is not a principal ideal.

3. Define I_K, the ideal class group of K.

   Let $K = \mathbb{Q}(\sqrt{34})$.

   (a) Determine which primes must be examined in order to determine I_K.

   (b) Use the Kummer–Dedekind theorem to factorize the principal ideals $(2), (3)$ and (5):

   $$(2) = P_2^2, \quad (3) = P_3 Q_3, \quad (5) = P_5 Q_5.$$

   (c) Use the equation $N_K(6 + \sqrt{34}) = 2$ to prove that $P_2 = (6 + \sqrt{34})$.

   (d) Let $\alpha = 7 + \sqrt{34}$. With a suitable choice of labelling, prove that $\alpha \in P_3$ and $\alpha \in P_5$ and deduce that

   $$P_3 P_5 = (\alpha).$$

   (e) Prove that $P_3^2 = (-5 + \sqrt{34})$.

(f) Given that $\eta = 35 + 6\sqrt{34}$ is the fundamental unit of K and that $N_K(\eta) = 1$, use Question 2(f) to prove that P_3 is not principal.

(g) Determine the structure of the class group I_K.

4. (a) If $a \in \mathbb{N}$, show that the polynomial $x^3 + ax - 1$ is irreducible over $\mathbb{Q}$ and has only one real root α.

(b) Prove that $\Delta(1, \alpha, \alpha^2) = -(4a^3 + 27)$.

(c) Suppose that $4a^3 + 27$ is squarefree. Explain why $D_K = -(4a^3 + 27)$ and prove that the fundamental unit η satisfies $\eta > a$.

(d) Observe that α is a unit between 0 and 1. Prove that $a < \alpha^{-1} < a + 1$ and deduce that $\eta = \alpha^{-1}$ if $a \geq 2$.

5. (a) Let $K = \mathbb{Q}(\sqrt{-5})$. Prove that the ideal $(3, 1 + \sqrt{-5})$ is not principal.

(b) Let $K = \mathbb{Q}(i)$.

(1) List the units of O_K.

(2) If $y = 2n + 1$, $n \in \mathbb{Z}$, prove that $1 + i$ divides $y \pm i$. Assuming that $\mathbb{Z}[i]$ is a UFD, prove that

$$\gcd\left(\frac{y+i}{1+i}, \frac{y-i}{1+i}\right) = 1.$$