MP473 Examination, November 2000

Time: 3 hours

Answer all questions

1. (a) Explain what is meant by the statements: (i) K *is an algebraic number field of degree* n, (ii) $\omega_1, \dots, \omega_n$ *is an integral basis for* K, (iii) U_K *is the unit group of* O_K, (iv) D_K, *the discriminant of* K, (v) $T_K(\alpha)$, the *trace* of α (vi) $N_K(\alpha)$, the *norm* of α.

   (b) Write down integral bases and discriminants for $\mathbb{Q}(\sqrt{d})$, when $d = -5$ and $d = 29$. Also describe explicitly the unit group U_K in each case.

   (c) Give an example in $K = \mathbb{Q}(i)$, where α and β belong to O_K, $N_K(\alpha)$ divides $N_K(\beta)$ in $\mathbb{Z}$, but α does not divide β in O_K.

2. Let $f(x) = x^4 + 2x^2 - 2 \in \mathbb{Q}[x]$..

   (a) Prove that $f(x) = x^4 + 2x^2 - 2$ is irreducible in $\mathbb{Q}[x]$.

   (b) Let $\theta^4 + 2\theta^2 - 2 = 0$ and let $K = \mathbb{Q}(\theta)$.

      (i) Prove that $2/\theta^2 \in O_K$ and that $2/\theta^4 \in U_K$. Also prove that $2/\theta^4 = 3 + \theta^2$.

      (ii) Verify that $3 = (1+\theta^2)^2$ and explain what this tells us about D_K.

      (iii) Given that $\Delta_K(1, \theta, \theta^2, \theta^3) = -2^9 3^2$, explain why $D_K = -2^9 3^2$.

3. (a) If I is an ideal of O_K, define $N(I)$, the *norm* of I.

   (b) If A and B are ideals of O_K and $A + B = \{a + b | a \in A, b \in B\}$, prove that $A + B$ is an ideal and that $A + B = \gcd(A, B)$.

   (c) If $K = \mathbb{Q}(\sqrt{-5})$ and $A = (2, 1 + \sqrt{-5})$, prove directly from the definition, without appealing to the Kummer–Dedekind theorem, that $N_K(A) = 2$.

   (d) Let I be an ideal of O_K and let $\alpha \in I$ satisfy

   $$N(I) = |N_K(\alpha)|.$$

   Prove that $I = (\alpha)$.

   (e) Let K be a real quadratic field and suppose that the fundamental unit η of K satisfies $N_K(\eta) = 1$. Let I be an ideal of O_K with the property that

   $$I^2 = (\alpha),$$

   where $N_K(\alpha) < 0$. Prove that I is not a principal ideal.

4. Let $K = \mathbb{Q}(\sqrt{-17})$.

   (a) Show that 2 and 3 are the only primes which must be examined in order to determine the ideal class group I_K.

   (b) Let $\omega = \sqrt{-17}$. Use the Kummer–Dedekind theorem to factorise (2) and (3): $(2) = Q^2$, $\quad (3) = PR$, where

   $$Q = (2, 1 + \omega),\ P = (3, 1 + \omega),\ R = (3, -1 + \omega).$$

(c) Prove that $P^2 = (9, 1+\omega)$ and $P^4 = (8-\omega)$.

(d) Verify that $(1-\omega)P^2 = (9)Q$.

(e) Explain why P^2 is not principal and I_K is cyclic of order 4.

5. (a) Define the term *UFD* in the context of the integral domain O_K. If O_K is a UFD and α, β, γ are non–zero integers in O_K with $\gcd(\alpha, \beta) = 1$ and satisfying

$$\alpha\beta = \gamma^2,$$

what can be said of α and β?

(b) Let x, y and z be rational integers satisfying $\gcd(x, y) = 1$ and

$$x^2 + y^2 = z^2. \qquad (1)$$

(i) Prove that x and y cannot both be odd.

(ii) If x is odd and y is even, Prove that $\gcd(x+iy, x-iy) = 1$ in $\mathbb{Z}[i]$.

(iii) By rewriting equation (1) as

$$(x+iy)(x-iy) = z^2,$$

use (a) and (b)(ii) to deduce that $x = a^2 - b^2$, $y = 2ab$, where a and b are relatively prime integers with one of a and b even, the other odd. (NB. The units of $\mathbb{Z}[i]$ are $\pm 1, \pm i$.)