PROBLEMS, Sheet 7, MP313, Semester 2, 1999.

1. Prove that if $n = a_0 + a_1 p + \cdots + a_s p^s$ is the base p expansion of n, $0 \leq a_i \leq p-1$, then defining $S_n = a_0 + \cdots + a_s$, we have $|n!|_p = p^{-t}$, where
$$t = \frac{n - S_n}{p-1}.$$

2. Show that the sequence $a_1 = 4, a_2 = 34, a_3 = 334, \cdots$ converges to $2/3$ in $\mathbb{Q}_5$. (Hint: Consider $3a_1, 3a_2, 3a_3, \cdots$.)

3. Find the first 3 digits of the square root a of 2 in $\hat{\mathbb{Z}}_7$ which satisfies $a \equiv 3 \pmod{7}$.

4. Find the first 3 digits of the fourth root a of 1 in $\hat{\mathbb{Z}}_5$ which satisfies $a \equiv 2 \pmod{5}$.

5. Prove that $x^2 + x + 223$ has a unique root a in $\hat{\mathbb{Z}}_3$ satisfying $a \equiv 4 \pmod{243}$. Find the first four digits of 3–adic expansion of a.

6. Let $p \equiv 2 \pmod{3}$. If a is an integer not divisible by p, show there is an $x \in \hat{\mathbb{Z}}_p$ with $x^3 = a$.

7. Let $a \in \mathbb{Z}$, $0 \leq a \leq p-1$. Prove that $\hat{\mathbb{Z}}_p$ always contains a unique solution to $x^p = x$, with $x \equiv a \pmod{p}$. (These are called *Teichmüller representatives.*)

8. Let $\alpha \in \hat{\mathbb{Z}}_p$. Prove that $\alpha^{p^M} \equiv \alpha^{p^{M-1}} \pmod{p^M}$ for $M \geq 1$ and deduce that the sequence $\{\alpha^{p^M}\}$ approaches a limit in $\hat{\mathbb{Z}}_p$ which is in fact the Teichmüller representative congruent to $\alpha \pmod{p}$.

9. Find the 2–adic expansion of $2/3$, the 7–adic expansion of $-1/6$ and the 13–adic expansion of $-9/16$.

10. Show that the mapping $f : \hat{\mathbb{Z}}_p \to \hat{\mathbb{Z}}_{p^2}$, given by the following formula is well–defined:
$$f(\{[x_n]\}) = \{[y_n]\}, \quad [y_n] = [x_{2n}] \in \mathbb{Z}_{p^{2n}}$$
and is an isomorphism between $\hat{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_{p^2}$, p a prime.

11. Use the Chinese remainder theorem to construct an isomorphism between $\hat{\mathbb{Z}}_{mn}$ and $\hat{\mathbb{Z}}_m \times \hat{\mathbb{Z}}_n$ if $\gcd(m,n) = 1$ and $m > 1, n > 1$. Also prove that $\hat{\mathbb{Z}}_{mn}$ is not an integral domain.