PROBLEMS, Sheet 6, MP313, Semester 2, 1999.

1. If p_n/q_n is the n–th convergent to α, prove that $q_n \geq F_{n+1}$.

2. If x_n denotes the n–th complete quotient to α, prove that

$$x_0 = \frac{p_n x_{n+1} + p_{n-1}}{q_n x_{n+1} + q_{n-1}}, \text{ if } n \geq 0.$$

3. Find the simple continued fraction for $\alpha = \frac{3+\sqrt{7}}{2}$.

4. If $x = [a_0, a_1, \ldots, a_k, 2a_0]$, where $a_1, \ldots, a_k$ is a symmetric string of positive integers, prove that x^2 is rational.

5. Prove that $\sqrt{d^2-1} = [d-1, \overline{1, 2d-2}]$ if $d > 1$.

6. If $d > 1$ is an odd integer, prove that

$$\sqrt{d^2+4} = [d; \overline{(d-1)/2, 1, 1, (d-1)/2, 2d}].$$

7. Prove that the continued fraction of $\sqrt{d}$ has period length 1 if and only if $d = a^2 + 1$, $a \geq 1$.

8. Let $d = a^2 + b$, where $a, b \in \mathbb{N}$, $b > 1$ and $b | 2a$. Prove that $[\sqrt{d}] = a$ and that $\sqrt{d}$ has the continued fraction expansion

$$\sqrt{d} = [a, \overline{\tfrac{2a}{b}, 2a}].$$

   Hence, or otherwise, derive the continued fraction expansion for $\sqrt{D^2 - D}$, when $D > 2$ is a positive integer.

   Conversely, if the continued fraction expansion of $\sqrt{d}$ has period length 2, show that $d = a^2 + b$, where $a, b \in \mathbb{N}$, $b > 1$ and $b | 2a$.

9. (H.J.S. Smith 1877) Use the equation

$$x_{n+1} = -(p_{n-1} - x q_{n-1})/(p_n - x_0 q_n)$$

   to prove that

$$x_1 \cdots x_{n+1} = \frac{(-1)^{n+1}}{p_n - x_0 q_n}. \tag{1}$$

Deduce that if $x_0 = [\sqrt{d}] + \sqrt{d}$ and $k+1$ is the period length of the simple continued fraction for x_0, then

$$x_0 \cdots x_k = p_k + q_k\sqrt{d}.$$

Illustrate with $d = 7$.

10. Let (x_1, y_1) and (x_2, y_2) be integer solutions of $x^2 - dy^2 = \pm 1$ such that $1 < x_1 + y_1\sqrt{d} < x_2 + y_2\sqrt{d}$. Prove that $x_1 < x_2$ and $y_1 < y_2$ either by using the result that $1 < x + y\sqrt{d}$ and $x^2 - dy^2 = \pm 1, x, y \in \mathbb{Z}$, implies $x > 0$ and $y > 0$, or otherwise.

11. Use question 9 to show that if $x_0 = \frac{P_0+\sqrt{d}}{Q_0}$, then

$$Q_0 p_n^2 - 2P_0 p_n q_n + \frac{P_0^2 - d}{Q_0} q_n^2 = (-1)^{n+1} Q_{n+1} \tag{2}$$

for $n \geq 0$.

(Hint: Apply σ to both sides of equation (1).)

Remark: If $x_0 = \sqrt{d}$, equation (2) reduces to $p_n^2 - dq_n^2 = (-1)^{n+1}Q_{n+1}$, while if $x_0 = \frac{\sqrt{d}-1}{2}$ and $d \equiv 1 \pmod 4$, equation (2) reduces to

$$p_n^2 + p_n q_n + \frac{1-d}{4} q_n^2 = (-1)^{n+1} \frac{Q_{n+1}}{2}.$$

12. Let d be a positive non–square integer, $\lambda = \lfloor \sqrt{d} \rfloor$ and let

$$\sqrt{d} = [\lambda,\ \overline{\lambda_1, \ldots, \lambda_k,\ 2\lambda}]$$

have period $k+1$. Writing $x_i = \frac{P_i+\sqrt{d}}{Q_i}$, show that x_{k+1} is the first complete quotient with $Q_i = 1$.

(Hint: Use the fact that x_i is reduced for $i \geq 1$, to show that $Q_i = 1 \Rightarrow P_i = \lambda \Rightarrow i \equiv 0 \pmod{k+1}$.)

Illustrate with $d = 47$.

13. Let $d > 1$ be a positive non–square rational and $\lambda = \lfloor \frac{\sqrt{d}-1}{2} \rfloor$.

(a) Show that $\alpha = \lambda + \frac{1+\sqrt{d}}{2}$ is reduced and that

$$\frac{\sqrt{d}-1}{2} = [\lambda, \overline{\lambda_1, \ldots, \lambda_k, 2\lambda+1}],$$

where the sequence $\lambda_1, \ldots, \lambda_k$ is symmetric.

(b) Let $d \in \mathbb{N}, d \equiv 1 \pmod 4$. Writing $x_i = \frac{P_i+\sqrt{d}}{Q_i}$, show that x_{k+1} is the first complete quotient with $Q_i = 2$.

(Hint: Use the fact that x_i is reduced for $i \geq 1$, to show that $Q_i = 2 \Rightarrow P_i = 2\lambda + 1 \Rightarrow i \equiv 0 \pmod{k+1}$.)

Illustrate with $d = 13$ and 93.

Remark. If $d > 1$ is a squarefree positive integer, this algorithm is used in **CALC** to find the fundamental unit $\eta_1 = p_k + q_k\omega$, of $\mathbb{Q}(\sqrt{d})$ if $d \equiv 1 \pmod 4$ and $\omega = \frac{1+\sqrt{d}}{2}$.

η_1 is the smallest unit > 1 in the ring $\mathbb{Z}[\omega]$ of *integers* of the form $a + b\omega$, $a, b \in \mathbb{Z}$. Here p_k/q_k satisfies

$$p_k^2 + p_k q_k + \frac{1-d}{4} q_k^2 = (-1)^{k+1}.$$

and is the first convergent to $\frac{\sqrt{d}-1}{2}$ for which $Q_{k+1} = 2$. Equivalently $(x, y) = (2p_k + q_k, q_k)$ is the solution in positive integers x, y, with smallest x, of the Pell–like equation $x^2 - dy^2 = \pm 4$.

If $d \equiv 2 \pmod 4$ or $d \equiv 3 \pmod 4$, the relevant ring of *integers* is $\mathbb{Z}[\sqrt{d}]$, the set of numbers $a + b\sqrt{d}, a, b \in \mathbb{Z}$. The fundamental unit $\eta_0 = p_k + q_k\sqrt{d}$, where p_k/q_k satisfies $p_k^2 - dq_k^2 = (-1)^{k+1}$ and is the first convergent to $\sqrt{d}$ for which $Q_{k+1} = 1$.

In both cases all units are given by $\pm\eta_1^n, \pm\eta_0^n, n \in \mathbb{Z}$, respectively.