4.3 Uniqueness of the Jordan form

Let β be a basis for V for which $[T]_\beta^\beta$ is in Jordan canonical form

$$J = J_{e_1}(\lambda_1) \oplus \cdots J_{e_s}(\lambda_s).$$

If we change the order of the basis vectors in β, we produce a corresponding change in the order of the elementary Jordan matrices. It is customary to assume our Jordan forms arranged so as to group together into a block those elementary Jordan matrices having the same eigenvalue c_i:

$$J = J_1 \oplus \cdots \oplus J_t,$$

where

$$J_i = \bigoplus_{j=1}^{\gamma_i} J_{e_{ij}}(c_i).$$

Moreover within this i–th block J_i, we assume the sizes $e_{i1}, \ldots, e_{i\gamma_i}$ of the elementary Jordan matrices decrease monotonically:

$$e_{i1} \geq \ldots \geq e_{i\gamma_i}.$$

We prove that with this convention, the above sequence is uniquely determined by T and the eigenvalue c_i.

We next observe that

$$ch_T = ch_J = \prod_{i=1}^{t} ch_{J_i} = \prod_{i=1}^{t}\prod_{j=1}^{\gamma_i}(x - c_i)^{e_{ij}} = \prod_{i=1}^{t}(x - c_i)^{e_{i1}+\cdots+e_{i\gamma_i}}.$$

Hence $c_1, \ldots, c_t$ are determined as the distinct eigenvalues of T.

DEFINITION 4.2

The numbers $e_{i1}, \ldots, e_{i\gamma_i}$, $1 \leq i \leq t$, are called the Segre *characteristic of T, while the numbers $\nu_{1,x-c_i}, \ldots, \nu_{b_i,x-c_i}$, $1 \leq i \leq t$ are called the* Weyr *characteristic of T.*

The polynomials $(x - c_i)^{e_{ij}}$ are called the elementary divisors *of T.*

LEMMA 4.1

Let

$$A = J_e(0) = \begin{bmatrix} 0 & 0 & & & 0 \\ 1 & 0 & \cdots & & \\ 0 & 1 & & & \\ & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 1 & 0 \end{bmatrix}.$$

Then

$$\nu(A^h) = \begin{cases} h & \text{if } 1 \leq h \leq e-1, \\ e & \text{if } e \leq h. \end{cases}$$

Proof. A^h has 1 on the h–th sub–diagonal, 0 elsewhere, if $1 \leq h \leq e-1$, whereas $A^h = 0$ if $h \geq e$.

Consequently

$$\nu(A^h) - \nu(A^{h-1}) = \begin{cases} 1 & \text{if } 1 \leq h \leq e, \\ 0 & \text{if } e < h. \end{cases}$$

We now can prove that the sequence $e_{i1} \geq \ldots e_{i\gamma_i}$ is determined uniquely by T and the eigenvalue c_i.

Let $p_k = x - c_k$ and

$$A = [T]_\beta^\beta = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{\gamma_i} J_{e_{ij}}(c_i).$$

Then

$$\begin{aligned} \nu(p_k^h(T)) &= \nu(p_k^h(A)) \\ &= \nu(\bigoplus_{i=1}^{t} \bigoplus_{j=1}^{\gamma_i} p_k^h(J_{e_{ij}}(c_i))) \\ &= \nu(\bigoplus_{i=1}^{t} \bigoplus_{j=1}^{\gamma_i} J_{e_{ij}}^h(c_i - c_k)) \\ &= \sum_{i=1}^{t} \sum_{j=1}^{\gamma_i} \nu(J_{e_{ij}}^h(c_i - c_k)), \end{aligned}$$

where we have used the fact that

$$p_k(J_{e_{ij}}(c_i)) = J_{e_{ij}}(c_i) - c_k I_n = J_{e_{ij}}(c_i - c_k).$$

However $J_{e_{ij}}(c_i - c_k)$ is a non–singular matrix if $i \neq k$, so

$$\nu(J_{e_{ij}}^h(c_i - c_k)) = 0$$

if $i \neq k$. Hence

$$\nu(p_k^h(T)) = \sum_{j=1}^{\gamma_k} \nu(J_{e_{kj}}^h(0)).$$

Hence

$$\begin{aligned}\nu_{h,\,x-c_k} = \nu(p_k^h(T)) - \nu(p_k^{h-1}(T)) &= \sum_{j=1}^{\gamma_k} \Big(\nu(J_{e_{kj}}^h(0)) - \nu(J_{e_{kj}}^{h-1}(0))\Big) \\ &= \sum_{\substack{j=1 \\ h \le e_{kj}}}^{\gamma_k} 1.\end{aligned}$$

Consequently $\nu_{h,\,x-c_k} - \nu_{h+1,\,x-c_k}$ is the number of e_{kj} which are equal to h. Hence by taking $h = 1, \ldots,$ we see that the sequence $e_{k1}, \ldots, e_{k\gamma_k}$ is determined by T and c_k and is in fact the contribution of the eigenvalue c_k to the Segre characteristic of T .

REMARK. If A and B are similar matrices over F, then $B = P^{-1}AP$ say. Also A and B have the same characteristic polynomials. Then if c_k is an eigenvalue of A and B and $p_k = x - c_k$, we have

$$p_k^h(T_B) = p_k^h(B) = P^{-1}p_k^h(A)P = P^{-1}p_k^h(T_A)P$$

and hence

$$\nu(p_k^h(T_B)) = \nu(p_k^h(T_A))$$

for all $h \geq 1$.

Consequently the Weyr characteristics of T_A and T_B will be identical. Hence the corresponding dot diagrams and so the Segre characteristics will also be identical. Hence T_A and T_B have the same Jordan form.

EXAMPLE 4.2

Let $A = J_2(0) \oplus J_2(0)$ and $B = J_2(0) \oplus J_1(0) \oplus J_1(0)$. Then

$$ch_A = ch_B = x^4 \quad \text{and} \quad m_A = m_B = x^2.$$

However A is not similar to B. For both matrices are in Jordan form and the Segre characteristics for T_A and T_B are 2, 2 and 2, 1, 1, respectively.

EXERCISE List all possible Jordan canonical forms of 2×2 and 3×3 matrices and deduce that if A and B have the same characteristic and same minimum polynomials, then A and B are similar if A and B are 2×2 or 3×3.

REMARK. Of course if A and B have the same Jordan canonical form, then A and B are similar.