4.10.3 A real algorithm for finding the real Jordan form

Referring to the last example, if we write $Z = \begin{bmatrix} A & I_4 \\ -I_4 & A \end{bmatrix}$, then

$$\begin{aligned} Z\left[\frac{X_{11}}{Y_{11}}\right] &= \left[\frac{X_{12}}{Y_{12}}\right], \\ Z\left[\frac{-Y_{11}}{X_{11}}\right] &= \left[\frac{-Y_{12}}{X_{12}}\right], \\ Z\left[\frac{X_{12}}{Y_{12}}\right] &= \left[\frac{0}{0}\right], \\ Z\left[\frac{-Y_{12}}{X_{12}}\right] &= \left[\frac{0}{0}\right]. \end{aligned}$$

Then the vectors

$$\left[\frac{X_{11}}{Y_{11}}\right], \left[\frac{-Y_{11}}{X_{11}}\right], Z\left[\frac{X_{11}}{Y_{11}}\right], Z\left[\frac{-Y_{11}}{X_{11}}\right]$$

actually form an $\mathbb{R}$–basis for $N(Z)$. This leads to a method for finding the real Jordan canonical form using real matrices. (I am indebted to Dr. B.D. Jones for introducing me to the Z matrix approach.)

More generally, we observe that a collection of equations of the form

$$\begin{aligned} AX_{ij1} &= a_iX_{ij1} - b_iY_{ij1} + X_{ij2} \\ AY_{ij1} &= b_iX_{ij1} + a_iY_{ij1} + Y_{ij2} \\ &\vdots \\ AX_{ije_{ij}} &= a_iX_{ije_{ij}} - b_iY_{ije_{ij}} \\ AY_{ije_{ij}} &= b_iX_{ije_{ij}} + a_iY_{ije_{ij}} \end{aligned}$$

can be written concisely in real matrix form, giving rise to an elementary Jordan basis corresponding to an elementary divisor $x^{e_{ij}}$ for the following real matrix: Let

$$Z_i = \begin{bmatrix} A - a_iI_n & b_iI_n \\ -b_iI_n & A - a_iI_n \end{bmatrix}.$$

Then

$$\begin{aligned} Z_i\left[\frac{X_{ij1}}{Y_{ij1}}\right] &= \left[\frac{X_{ij2}}{Y_{ij2}}\right] \\ &\vdots \\ Z_i\left[\frac{X_{ije_{ij}}}{Y_{ije_{ij}}}\right] &= \left[\frac{0}{0}\right]. \end{aligned}$$

LEMMA 4.2

If V is a $\mathbb{C}$–vector space with basis $v_1, \dots, v_n$, then V is also an $\mathbb{R}$–vector space with basis

$$v_1,\ iv_1, \dots, v_n,\ iv_n.$$

Hence

$$\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V.$$

DEFINITION 4.8

Let $A \in M_{n\times n}(\mathbb{R})$ and $c = a + ib$ be a complex eigenvalue of A with $b \neq 0$. Let $Z \in M_{2n\times 2n}(\mathbb{R})$ be defined by

$$Z = \begin{bmatrix} A - aI_n & bI_n \\ -bI_n & A - aI_n \end{bmatrix} = (A - aI_n) \otimes I_n - I_n \otimes (bJ).$$

Also let $p = x - c$.

LEMMA 4.3

Let $\Phi : V_{2n}(\mathbb{R}) \to V_n(\mathbb{C})$ be the mapping defined by

$$\Phi\left(\begin{bmatrix} X \\ \hline Y \end{bmatrix}\right) = X + iY, \quad X,\, Y \in V_n(\mathbb{R}).$$

Then

(i) Φ *is an* $\mathbb{R}$– *isomorphism;*

(ii) $\Phi\left(\left[\frac{-Y}{X}\right]\right) = i(X + iY)$;

(iii) $\Phi\left(Z^h \left[\frac{X}{Y}\right]\right) = p^h(A)(X + iY)$;

(iv) $\Phi\left(Z^h \left[\frac{-Y}{X}\right]\right) = ip^h(A)(X + iY)$;

(v) Φ *maps* $N(Z^h)$ *onto* $N(p^h(A)$;

COROLLARY 4.4

If

$$p^{e_1-1}(A)(X_1 + iY_1), \dots, p^{e_\gamma-1}(A)(X_\gamma + iY_\gamma)$$

form a $\mathbb{C}$–basis for $N(p(A))$, then

$$Z^{e_1-1}\begin{bmatrix} X_1 \\ \hline Y_1 \end{bmatrix},\ Z^{e_1-1}\begin{bmatrix} -Y_1 \\ \hline X_1 \end{bmatrix}, \dots, Z^{e_\gamma-1}\begin{bmatrix} X_\gamma \\ \hline Y_\gamma \end{bmatrix},\ Z^{e_\gamma-1}\begin{bmatrix} -Y_\gamma \\ \hline X_\gamma \end{bmatrix}$$

form an $\mathbb{R}$–basis for $N(Z)$ and conversely.

Remark: Consequently the dot diagram for the eigenvalue 0 for the matrix Z has the same height as that for the eigenvalue c of A, with each row expanded to twice the length.

To find suitable vectors $X_1, Y_1, \ldots, X_\gamma, Y_\gamma$, we employ the usual algorithm for finding the Jordan blocks corresponding to the eigenvalue 0 of the matrix Z, with the extra proviso that we always ensure that the basis for $N_{h,x}$ is chosen to have the form

$$Z^{h-1}\left[\frac{X_1}{Y_1}\right],\ Z^{h-1}\left[\frac{-Y_1}{X_1}\right],\ldots,Z^{h-1}\left[\frac{X_r}{Y_r}\right],\ Z^{h-1}\left[\frac{-Y_r}{X_r}\right],$$

where $r = (\,\text{nullity}\, Z^h - \text{ nullity}\, Z^{h-1})/2$.

This can be ensured by extending a spanning family for $N(Z^h)$:

$$\left[\frac{X_1}{Y_1}\right],\ldots,\left[\frac{X_{\nu(Z^h)}}{Y_{\nu(Z^h)}}\right]$$

to the form

$$\left[\frac{X_1}{Y_1}\right],\ \left[\frac{-Y_1}{X_1}\right],\ldots,\left[\frac{X_{\nu(Z^h)}}{Y_{\nu(Z^h)}}\right],\ \left[\frac{-Y_{\nu(Z^h)}}{X_{\nu(Z^h)}}\right].$$

EXAMPLE 4.8
$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ -2 & -1 & -1 & -1 \end{bmatrix} \in M_{4\times 4}(\mathbb{R})$ *has* $m_A = (x^2+1)^2$. *Find a real non–singular matrix* P *such that* $P^{-1}AP$ *is in real Jordan form.*
Solution:

$$Z = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 \end{bmatrix}$$

$$\textit{basis for } N(Z^2) : \begin{bmatrix} 1 & 1 & 1/2 & 1/2 \\ -2 & -1/2 & 0 & -1/2 \\ 2 & 1/2 & 1 & 3/2 \\ -2 & -3/2 & -2 & -3/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\textit{blown–up basis for } N(Z^2) : \begin{bmatrix} 1 & -1 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ -2 & 0 & -1/2 & -1 & 0 & 0 & -1/2 & 0 \\ 2 & 0 & 1/2 & 0 & 1 & -1 & 3/2 & 0 \\ -2 & 0 & -3/2 & 0 & -2 & 0 & -3/2 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & -2 & 1 & -1/2 & 0 & 0 & 0 & -1/2 \\ 0 & 2 & 0 & 1/2 & 1 & 1 & 0 & 3/2 \\ 0 & -2 & 0 & -3/2 & 0 & -2 & 1 & -3/2 \end{bmatrix}$$

$$\rightarrow \textit{ left–to–right basis for } N(Z^2) : \begin{bmatrix} 1 & -1 & 1 & 0 \\ -2 & 0 & -1/2 & -1 \\ 2 & 0 & 1/2 & 0 \\ -2 & 0 & -3/2 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1/2 \\ 0 & 2 & 0 & 1/2 \\ 0 & -2 & 0 & -3/2 \end{bmatrix}$$

We then derive a spanning family for $N_{2,x}$:

$$Z \times \textit{ basis matrix} = \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix} \quad \rightarrow \quad \textit{basis for } N_{2,x} :$$

$$\begin{bmatrix} 1/2 & 0 \\ -1/2 & -1/2 \\ 1/2 & 1/2 \\ -1/2 & -1/2 \\ 0 & 1/2 \\ 1/2 & -1/2 \\ -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Consequently we read off that $Z\left[\frac{X_{11}}{Y_{11}}\right] = \left[\frac{X_{12}}{Y_{12}}\right]$ *is a basis for* $N_{2,x} = N_{1,x} = N(Z)$*. where*

$$P = [X_{11}|Y_{11}|X_{12}|Y_{12}] = \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ -1/2 & 1 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & -1/2 \\ -3/2 & 0 & -1/2 & 1/2 \end{bmatrix}.$$

Then

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

which is in real Jordan form.