4.10 The Real Jordan Form

4.10.1 Motivation

If A is a real $n \times n$ matrix, the characteristic polynomial of A will in general have real roots and complex roots, the latter occurring in complex pairs. In this section we show how to derive a canonical form B for A which has real entries. It turns out that there is a simple formula for e^B and this is useful in solving $\dot{X} = AX$, as it allows one to directly express the complete solution of the system of differential equations in terms of real exponentials and sines and cosines.

We first introduce a real analogue of $J_n(a+ib)$. It's the matrix $K_n(a, b) \in M_{2n\times 2n}(\mathbb{R})$ defined as follows:

Let $D = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = aI_2 + bJ$ where $J^2 = -I_2$ (J is a matrix version of $i = \sqrt{-1}$, while D corresponds to the complex number $a + ib$) then

$$\begin{aligned} e^D &= e^{aI_2+bJ} \\ &= e^{aI_2}e^{bJ} \\ &= e^a I_2 \left[I_2 + \frac{bJ}{1!} + \frac{(bJ)^2}{2!} + \cdots \right] \\ &= e^a \left[\left\{ I_2 - \frac{b^2}{2!}I_2 + \frac{b^4}{4!}I_2 + \cdots \right\} + \left\{ \frac{b}{1!}J - \frac{b^3}{3!}J + \cdots \right\} \right] \\ &= e^a \left[(\cos b)I_2 + (\sin b)J \right] \\ &= e^a \begin{bmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{bmatrix}. \end{aligned}$$

Replacing a and b by ta and tb, where $t \in \mathbb{R}$, gives

$$e^{tD} = e^{at} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}.$$

DEFINITION 4.7

Let a and b be real numbers and $K_n(a, b) \in M_{2n\times 2n}(\mathbb{R})$ be defined by

$$K_n(a, b) = \left[\begin{array}{c|c|c} D & 0 & \cdots \\ \hline I_2 & D & \\ \hline 0 & I_2 & \\ & & \ddots \\ & & & D \end{array} \right]$$

where $D = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. *Then it is easy to prove that*

$$e^{K_n(a,b)} = \left[\begin{array}{c|c|ccc} e^D & 0 & \cdots & & \\ \hline e^D/1! & e^D & & & \\ \hline e^D/2! & e^D/1! & \ddots & & \\ \vdots & & \ddots & \ddots & \\ e^D/(n-1)\ ! & \cdots & \cdots & e^D/1! & e^D \end{array}\right].$$

EXAMPLE 4.6

$$K_2(0,\,1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

and

$$e^{tK_2(0,\,1)} = \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ t\cos t & t\sin t & \cos t & \sin t \\ -t\sin t & t\cos t & -\sin t & \cos t \end{bmatrix}.$$

4.10.2 Determining the real Jordan form

If $A = [a_{ij}]$ is a complex matrix, let $\overline{A} = [\overline{a}_{ij}]$. Then

1.
$$\overline{A \pm B} = \overline{A} \pm \overline{B},\ \overline{cA} = \overline{c}\overline{A}\ c \in \mathbb{C},\ \overline{AB} = \overline{A} \cdot \overline{B}.$$

2. If $A \in M_{n\times n}(\mathbb{R})$ and $a_0, \ldots, a_r \in \mathbb{C}$, then
$$\overline{a_0 I_n + \cdots a_r A^r} = \overline{a}_0 I_n + \cdots + \overline{a}_r A^r.$$

3. If W is a subspace of $V_n(\mathbb{C})$, then so is $\overline{W} = \{\overline{w} | w \in W\}$.
   Moreover if $W = \langle w_1, \ldots, w_r\rangle$, then
$$\overline{W} = \langle \overline{w}_1, \ldots, \overline{w}_r\rangle.$$

4. If $w_1, \ldots, w_r$ are linearly independent vectors in $V_n(\mathbb{C})$, then so are $\overline{w}_1, \ldots, \overline{w}_r$. Hence if $w_1, \ldots, w_r$ form a basis for a subspace W, then $\overline{w}_1, \ldots, \overline{w}_r$ form a basis for $\overline{W}$.

5. Let A be a real $n \times n$ matrix and $c \in \mathbb{C}$. Then

(a)

$$W = N((A - cI_n)^h) \Rightarrow \overline{W} = N((A - \overline{c}I_n)^h).$$

(b)

$$W = W_1 \oplus \cdots \oplus W_r \Rightarrow \overline{W} = \overline{W}_1 \oplus \cdots \oplus \overline{W}_r.$$

(c)

$$W = C_{T_A,\, v} \Rightarrow \overline{W} = C_{T_A,\, \overline{v}}.$$

(d)

$$W = \bigoplus_{i=1}^{r} C_{T_A,\, v_i} \Rightarrow \overline{W} = \bigoplus_{i=1}^{r} C_{T_A,\, \overline{v}_i}.$$

(e)

$$m_{T_A,\, v} = (x - c)^e \Rightarrow m_{T_A,\, \overline{v}} = (x - \overline{c})^e.$$

Let $A \in M_{n\times n}(\mathbb{R})$. Then $m_A \in \mathbb{R}[x]$ and so any complex roots will occur in conjugate pairs.

Suppose that $c_1, \ldots, c_r$ are the distinct real eigenvalues and $c_{r+1}, \ldots, c_{r+s}$, $\bar{c}_{r+1}, \ldots, \bar{c}_{r+s}$ are the distinct non-real roots and

$$\begin{aligned} m_A &= (x - c_1)^{b_1} \ldots (x - c_r)^{b_r} (x - c_{r+1})^{b_{r+1}} \ldots (x - c_{r+s})^{b_{r+s}} \\ &\quad \times (x - \bar{c}_{r+1})^{b_{r+1}} \ldots (x - \bar{c}_{r+s})^{b_{r+s}}. \end{aligned}$$

For each complex eigenvalue c_i, $r+1 \leq i \leq r+s$, there exists a secondary decomposition

$$N(A - c_i I_n)^{b_i} = \bigoplus_{j=1}^{\gamma_i} C_{T_A, v_{ij}}, \quad m_{T_A,\, v_{ij}} = (x - c_i)^{e_{ij}}$$

Hence we have a corresponding secondary decomposition for the eigenvalue $\bar{c_i}$:

$$N(A - \bar{c}_i I_n)^{b_i} = \bigoplus_{j=1}^{\gamma_i} C_{T_A, \bar{v}_{ij}}, \quad m_{T_A,\, \bar{v}_{ij}} = (x - \bar{c}_i)^{e_{ij}}.$$

For brevity, let $c = c_i$, $v = v_{ij}$, $e = e_{ij}$. Let

$$P_1 = v, P_2 = (A - cI_n)P_1, \ \ldots, P_e = (A - cI_n)P_{e-1}$$

and

$$P_1 = X_1 + iY_1,\ P_2 = X_2 + iY_2,\ \ldots,\ P_e = X_e + iY_e;\ c = a + ib.$$

Then we have the following equations, posed in two different ways:

$$\begin{array}{rcl|rcl} AP_1 & = & cP_1 + P_2 & AX_1 & = & aX_1 - bY_1 + X_2 \\ & & & AY_1 & = & bX_1 + aY_1 + Y_2 \\ \vdots & & & \vdots & & \\ AP_e & = & cP_e & AX_e & = & aX_e - bY_e \\ & & & AY_e & = & bX_e + aY_e. \end{array}$$

In matrix terms we have

$$A[X_1|Y_1|X_2|Y_2|\cdots|X_e|Y_e] =$$

$$[X_1|Y_1|X_2|Y_2|\cdots|X_e|Y_e]\left[\begin{array}{cc|cc|c|cc} a & b & & & \cdots & & \\ -b & a & & & & & \\ \hline 1 & 0 & a & b & & & \\ 0 & 1 & -b & a & & & \\ \vdots & & & & \ddots & & \\ & & & & & a & b \\ 0 & & & & & -b & a \end{array}\right].$$

The large "real jordan form" matrix is the $2e \times 2e$ matrix $K_e(a, b)$.
Note: If $e = 1$, no I_2 block is present in this matrix.

The spaces $C_{T_A,\,v}$ and $C_{T_A,\,\bar{v}}$ are independent and have bases $P_1, \ldots, P_e$ and $\bar{P}_1, \ldots, \bar{P}_e$, respectively.

Consequently the vectors

$$P_1, \ldots, P_e,\ \bar{P}_1, \ldots, \bar{P}_e$$

form a basis for $C_{T_A,\,v} + C_{T_A,\,\bar{v}}$. It is then an easy exercise to deduce that the real vectors X_1, $Y_1, \ldots, X_e$, Y_e form a basis β for the T–invariant subspace

$$W = C_{T_A,\,v} + C_{T_A,\,\bar{v}}.$$

Writing $T = T_A$ for brevity, the above right hand batch of equations tells us that $[T_W]_\beta^\beta = K_e(a, b)$. There will be s such real bases corresponding to each of the complex eigenvalues $c_{r+1} \ldots, c_{r+s}$.

Joining together these bases with the real elementary Jordan bases arising from any real eigenvalues $c_1, \ldots, c_r$ gives a basis β for $V_n(\mathbb{C})$ such that if P is the non–singular real matrix formed by these basis vectors, then

$$P^{-1}AP = [T_A]_\beta^\beta = J \oplus K,$$

where

$$J = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{\gamma_i} J_{e_{ij}}(c_i), \quad K = \bigoplus_{i=r+1}^{r+s} \bigoplus_{j=1}^{\gamma_i} K_{e_{ij}}(a_i, b_i),$$

where $c_i = a_i + ib_i$ for $r + 1 \leq i \leq r + s$.

The matrix $J \oplus K$ is said to be in real Jordan canonical form.

EXAMPLE 4.7

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ -2 & -1 & -1 & -1 \end{bmatrix} \quad \text{so} \quad \begin{aligned} m_A &= (x^2+1)^2 \\ &= (x-i)^2(x+i)^2. \end{aligned}$$

Thus with $p_1 = x - i$, we have the dot diagram

$$\begin{array}{|c|l} \hline \cdot & N_{2,p_1} \\ \hline \cdot & N_{1,p_1} = N(A - iI_4). \\ \hline \end{array}$$

Thus we find an elementary Jordan basis for N_{1,p_1}:

$$X_{11} + iY_{11},\ (A - iI_4)(X_{11} + iY_{11}) = X_{12} + iY_{12}$$

yielding

$$\begin{aligned} AX_{11} &= -Y_{11} + X_{12} \\ AY_{11} &= X_{11} + Y_{12}. \end{aligned} \tag{22}$$

Now we know

$$\begin{aligned} & m_{T_A, X_{11}+iY_{11}} = (x-i)^2 \\ \Rightarrow\ & (A - iI_4)^2(X_{11} + iY_{11}) = 0 \\ \Rightarrow\ & (A - iI_4)(X_{12} + iY_{12}) = 0 \\ \Rightarrow\ & AX_{12} = -Y_{12} \\ & AY_{12} = X_{12}. \end{aligned} \tag{23}$$

Writing the four real equations (22) and (23) in matrix form, with

$$P = [X_{11}|Y_{11}|X_{12}|Y_{12}],$$

then P is non-singular and

$$P^{-1}AP = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right].$$

The numerical determination of P is left as a tutorial problem.