4.10 The Real Jordan Form

4.10.1 Motivation

If $A$ is a real $n \times n$ matrix, the characteristic polynomial of $A$ will in general have real roots and complex roots, the latter occurring in complex pairs. In this section we show how to derive a canonical form $B$ for $A$ which has real entries. It turns out that there is a simple formula for $e^B$ and this is useful in solving $\dot{X} = AX$, as it allows one to directly express the complete solution of the system of differential equations in terms of real exponentials and sines and cosines.

We first introduce a real analogue of $J_n(a+ib)$. It’s the matrix $K_n(a, b) \in M_{2n \times 2n}(\mathbb{R})$ defined as follows:

Let $D = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = aI_2 + bJ$ where $J^2 = -I_2$ ($J$ is a matrix version of $i = \sqrt{-1}$, while $D$ corresponds to the complex number $a + ib$) then

$$e^D = e^{aI_2 + bJ} = e^{aI_2} e^{bJ} = e^a I_2 \left[ I_2 + \frac{bJ}{1!} + \frac{(bJ)^2}{2!} + \cdots \right]$$

$$= e^a \left[ \left\{ I_2 - \frac{b^2}{2!} I_2 + \frac{b^4}{4!} I_2 + \cdots \right\} + \left\{ \frac{b}{1!} J - \frac{b^3}{3!} J + \cdots \right\} \right]$$

$$= e^a \left[ (\cos b) I_2 + (\sin b) J \right]$$

$$= e^a \begin{bmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{bmatrix}.$$  

Replacing $a$ and $b$ by $ta$ and $tb$, where $t \in \mathbb{R}$, gives

$$e^{tD} = e^{at} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}.$$ 

**DEFINITION 4.7**

Let $a$ and $b$ be real numbers and $K_n(a, b) \in M_{2n \times 2n}(\mathbb{R})$ be defined by

$$K_n(a, b) = \begin{bmatrix} D & 0 & \cdots \\ I_2 & D & \cdots \\ \vdots & \ddots & \ddots \\ 0 & I_2 & \cdots \\ & & & D \end{bmatrix}$$

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where \( D = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \). Then it is easy to prove that
\[
e^{K_n(a, b)} = \begin{bmatrix} e^D & 0 & \cdots \\ e^D/1! & e^D & \cdots \\ e^D/2! & e^D/1! & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ e^D/(n-1)! & \cdots & \cdots & e^D/1! & e^D \end{bmatrix}.
\]

**EXAMPLE 4.6**

\[
K_2(0, 1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}
\]

and
\[
e^{tK_2(0, 1)} = \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ t \cos t & t \sin t & \cos t & \sin t \\ -t \sin t & t \cos t & -\sin t & \cos t \end{bmatrix}.
\]

**4.10.2 Determining the real Jordan form**

If \( A = [a_{ij}] \) is a complex matrix, let \( \overline{A} = [\overline{a}_{ij}] \). Then

1. \( \overline{A \pm B} = \overline{A \pm B}, \overline{cA} = \overline{c}A \ c \in \mathbb{C}, \overline{AB} = \overline{A} \overline{B} \).

2. If \( A \in M_{n \times n}(\mathbb{R}) \) and \( a_0, \ldots, a_r \in \mathbb{C} \), then
\[
\overline{a_0I_n + \cdots + a_rA^r} = \overline{a_0I_n} + \cdots + \overline{a_rA^r}.
\]

3. If \( W \) is a subspace of \( V_n(\mathbb{C}) \), then so is \( \overline{W} = \{ \overline{w} | w \in W \} \).
Moreover if \( W = \langle w_1, \ldots, w_r \rangle \), then
\[
\overline{W} = \langle \overline{w_1}, \ldots, \overline{w_r} \rangle.
\]

4. If \( w_1, \ldots, w_r \) are linearly independent vectors in \( V_n(\mathbb{C}) \), then so are \( \overline{w_1}, \ldots, \overline{w_r} \). Hence if \( w_1, \ldots, w_r \) form a basis for a subspace \( W \), then \( \overline{w_1}, \ldots, \overline{w_r} \) form a basis for \( \overline{W} \).
5. Let $A$ be a real $n \times n$ matrix and $c \in \mathbb{C}$. Then

(a) \[ W = N((A - cI_n)^h) \Rightarrow \overline{W} = N((A - \overline{c}I_n)^h). \]

(b) \[ W = W_1 \oplus \cdots \oplus W_r \Rightarrow \overline{W} = \overline{W_1} \oplus \cdots \oplus \overline{W_r}. \]

(c) \[ W = C_{T_A, v} \Rightarrow \overline{W} = C_{T_A, \overline{v}}. \]

(d) \[ W = \bigoplus_{i=1}^r C_{T_A, v_i} \Rightarrow \overline{W} = \bigoplus_{i=1}^r C_{T_A, \overline{v}_i}. \]

(e) \[ m_{T_A, v} = (x - c)^e \Rightarrow \overline{m_{T_A, v}} = (x - \overline{c})^e. \]

Let $A \in M_{n \times n}(\mathbb{R})$. Then $m_A \in \mathbb{R}[x]$ and so any complex roots will occur in conjugate pairs.

Suppose that $c_1, \ldots, c_r$ are the distinct real eigenvalues and $c_{r+1}, \ldots, c_{r+s}, \overline{c}_{r+1}, \ldots, \overline{c}_{r+s}$ are the distinct non-real roots and

\[ m_A = (x - c_1)^{b_1} \cdots (x - c_r)^{b_r} (x - c_{r+1})^{b_{r+1}} \cdots (x - c_{r+s})^{b_{r+s}} \times (x - \overline{c}_{r+1})^{\overline{b}_{r+1}} \cdots (x - \overline{c}_{r+s})^{\overline{b}_{r+s}}. \]

For each complex eigenvalue $c_i$, $r+1 \leq i \leq r+s$, there exists a secondary decomposition

\[ N(A - c_i I_n)^{h_i} = \bigoplus_{j=1}^{\gamma_i} C_{T_A, v_{ij}}, \quad m_{T_A, v_{ij}} = (x - c_i)^{e_{ij}}. \]

Hence we have a corresponding secondary decomposition for the eigenvalue $\overline{c}_i$:

\[ N(A - \overline{c}_i I_n)^{b_i} = \bigoplus_{j=1}^{\gamma_i} C_{T_A, \overline{v}_{ij}}, \quad m_{T_A, \overline{v}_{ij}} = (x - \overline{c}_i)^{\overline{e}_{ij}}. \]
For brevity, let $c = c_i$, $v = v_{ij}$, $e = e_{ij}$. Let

$$P_1 = v, P_2 = (A - cI_n)P_1, \ldots, P_e = (A - cI_n)P_{e-1}$$

and

$$P_1 = X_1 + iY_1, P_2 = X_2 + iY_2, \ldots, P_e = X_e + iY_e; c = a + ib.$$  

Then we have the following equations, posed in two different ways:

$$AP_1 = cP_1 + P_2 \quad AX_1 = aX_1 - bY_1 + X_2$$

$$\vdots$$

$$AP_e = cP_e \quad AX_e = aX_e - bY_e$$

$$AY_1 = bX_1 + aY_1 + Y_2$$

In matrix terms we have

$$A[X_1|Y_1|X_2|Y_2| \cdots |X_e|Y_e] = \begin{bmatrix} a & b \\ -b & a \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \cdots$$

The large “real jordan form” matrix is the $2e \times 2e$ matrix $K_e(a, b)$.

**Note:** If $c = 1$, no $I_2$ block is present in this matrix.

The spaces $C_{T_A,v}$ and $C_{T_A,\bar{v}}$ are independent and have bases $P_1, \ldots, P_e$ and $P_1, \ldots, P_e$, respectively.

Consequently the vectors

$$P_1, \ldots, P_e, \bar{P}_1, \ldots, \bar{P}_e$$

form a basis for $C_{T_A,v} + C_{T_A,\bar{v}}$. It is then an easy exercise to deduce that the real vectors $X_1, Y_1, \ldots, X_e, Y_e$ form a basis $\beta$ for the $T$–invariant subspace

$$W = C_{T_A,v} + C_{T_A,\bar{v}}.$$  

Writing $T = T_A$ for brevity, the above right hand batch of equations tells us that $[T_W]^\beta = K_e(a, b)$. There will be $s$ such real bases corresponding to each of the complex eigenvalues $c_{r+1}, \ldots, c_{r+s}$.
Joining together these bases with the real elementary Jordan bases arising from any real eigenvalues $c_1, \ldots, c_r$ gives a basis $\beta$ for $V_n(\mathbb{C})$ such that if $P$ is the non-singular real matrix formed by these basis vectors, then

$$P^{-1}AP = [T_A]_\beta = J \oplus K,$$

where

$$J = \bigoplus_{i=1}^r \gamma_i J_{e_{ij}}(c_i), \quad K = \bigoplus_{i=r+1}^{r+s} \gamma_i K_{e_{ij}}(a_i, b_i),$$

where $c_i = a_i + ib_i$ for $r + 1 \leq i \leq r + s$.

The matrix $J \oplus K$ is said to be in real Jordan canonical form.

**EXAMPLE 4.7**

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ -2 & -1 & -1 & -1 \end{bmatrix}$$

so

$$m_A = (x^2 + 1)^2 = (x - i)^2(x + i)^2.$$ 

Thus with $p_1 = x - i$, we have the dot diagram

$\begin{array}{c} N_{2,p_1} \\ N_{1,p_1} = N(A - iI_4). \end{array}$

Thus we find an elementary Jordan basis for $N_{1,p_1}$:

$$X_{11} + iY_{11}, \quad (A - iI_4)(X_{11} + iY_{11}) = X_{12} + iY_{12}$$

yielding

$$AX_{11} = -Y_{11} + X_{12}$$

$$AY_{11} = X_{11} + Y_{12}. \quad (22)$$

Now we know

$$m_{T_A,X_{11}+iY_{11}} = (x - i)^2$$

$$\Rightarrow (A - iI_4)^2(X_{11} + iY_{11}) = 0$$

$$\Rightarrow (A - iI_4)(X_{12} + iY_{12}) = 0$$

$$\Rightarrow AX_{12} = -Y_{12}$$

$$AY_{12} = X_{12}. \quad (23)$$

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Writing the four real equations (22) and (23) in matrix form, with
\[ P = \begin{bmatrix} X_{11} & Y_{11} \mid X_{12} & Y_{12} \end{bmatrix}, \]
then \( P \) is non-singular and
\[ P^{-1} A P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}. \]

The numerical determination of \( P \) is left as a tutorial problem.