4.9 Markov matrices

DEFINITION 4.3
A real $n \times n$ matrix $A = [a_{ij}]$ is called a **Markov** *matrix, or* **row–stochastic matrix** *if*

(i) $a_{ij} \geq 0$ *for* $1 \leq i, j \leq n$*;*

(ii) $\sum_{j=1}^{n} a_{ij} = 1$ *for* $1 \leq i \leq n$*.*

Remark: (ii) is equivalent to $AJ_n = J_n$, where $J_n = [1, \ldots, 1]^t$. So 1 is always an eigenvalue of a Markov matrix.

EXERCISE 4.1
If A and B are $n \times n$ Markov matrices, prove that AB is also a Markov matrix.

THEOREM 4.9
Every eigenvalue λ of a Markov matrix satisfies $|\lambda| \leq 1$.

PROOF Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of A and $X \in V_n(\mathbb{C})$ is a corresponding eigenvector. Then

$$AX = \lambda X. \tag{13}$$

Let k be such that $|x_j| \leq |x_k|$, $\forall j$, $1 \leq j \leq n$. Then equating the k–th component of each side of equation (13) gives

$$\sum_{j=1}^{n} a_{kj} x_j = \lambda x_k. \tag{14}$$

Hence

$$|\lambda x_k| = |\lambda| \cdot |x_k| = |\sum_{j=1}^{n} a_{kj} x_j| \leq \sum_{j=1}^{n} a_{kj} |x_j| \tag{15}$$

$$\leq \sum_{j=1}^{n} a_{kj} |x_k| = |x_k|. \tag{16}$$

Hence $|\lambda| \leq 1$.

DEFINITION 4.4

A **positive Markov matrix** *is one with all positive elements (i.e. strictly greater than zero). For such a matrix A we may write "$A > 0$".*

THEOREM 4.10

If A is a positive Markov matrix, then 1 is the only eigenvalue of modulus 1. Moreover nullity $(A - I_n) = 1$.

PROOF Suppose $|\lambda| = 1$, $AX = \lambda X$, $X \in V_n(\mathbb{C})$, $X \neq 0$.

Then inequalities (15) and (16) reduce to

$$|x_k| = \Big|\sum_{j=1}^{n} a_{kj}x_j\Big| \leq \sum_{j=1}^{n} a_{kj}|x_j| \leq \sum_{j=1}^{n} a_{kj}|x_k| = |x_k|. \tag{17}$$

Then inequalities (17) and a sandwich principle, give

$$|x_j| = |x_k| \quad \text{for} \quad 1 \leq j \leq n. \tag{18}$$

Also, as equality holds in the triangle inequality section of inequalities (17), this forces all the complex numbers $a_{kj}x_j$ to lie in the same direction:

$$\begin{aligned} a_{kj}x_j &= t_j a_{kk} x_k, \ , t_j > 0, \ 1 \leq j \leq n, \\ x_j &= \tau_j x_k, \end{aligned}$$

where $\tau_j = (t_j a_{kk})/a_{kj} > 0$.

Then equation (18) implies $\tau_j = 1$ and hence $x_j = x_k$ for $1 \leq j \leq n$.

Consequently $X = x_k J_n$, thereby proving that $N(A - I_n) = \langle J_n \rangle$.

Finally, equation (14) implies

$$\sum_{j=1}^{n} a_{kj}x_j = \lambda x_k = \sum_{j=1}^{n} a_{kj}x_k = x_k,$$

so $\lambda = 1$.

COROLLARY 4.3

If A is a positive Markov matrix, then A^t has 1 as the only eigenvalue of modulus 1. Also nullity $(A^t - I_n) = 1$.

PROOF The eigenvalues of A^t are precisely the same as those of A, even up to multiplicities. For

$$\text{ch}_{A^t} = \det(xI_n - A^t) = \det(xI_n - A)^t = \det(xI_n - A) = \text{ch}_A.$$

Also $\nu(A^t - I_n) = \nu(A - I_n)^t = \nu(A - I_n) = 1$.

THEOREM 4.11

If A is a positive Markov matrix, then

(i) $(x-1)||m_A$;

(ii) $A^m \to B$, *where* $B = \begin{bmatrix} X^t \\ \hline \vdots \\ \hline X^t \end{bmatrix}$ *is a positive Markov matrix and where* X *is uniquely defined as the (positive) vector satisfying* $A^tX = X$ *whose components sum to* 1.

Remark: In view of part (i) and the equation $\nu(A - I_n) = 1$, it follows that $(x-1)||\operatorname{ch}_A$.

PROOF As $\nu(A - I_n) = 1$, the Jordan form of A has the form $J_b(1) \oplus K$, where $(x-1)^b||m_A$. Here K is the direct sum of all Jordan blocks corresponding to all the eigenvalues of A other than 1 and hence $K^m \to 0$.

Now suppose that $b > 1$; then $J_b(1)$ has size $b > 1$. Then $\exists P$ such that

$$\begin{aligned} P^{-1}AP &= J_b(1) \oplus K, \\ P^{-1}A^mP &= J_b^m(1) \oplus K^m. \end{aligned}$$

Hence the 2×1 element of $J_b^m(1)$ equals $\binom{m}{1} \to \infty$ as $m \to \infty$.

However the elements of A^m are ≤ 1, as A^m is a Markov matrix. Consequently the elements of $P^{-1}A^mP$ are bounded as $m \to \infty$. This contradiction proves that $b = 1$.

Hence $P^{-1}A^mP \to I_1 \oplus 0$ and $A^m \to P(I_1 \oplus 0)P^{-1} = B$.

We see that $\operatorname{rank} B = \operatorname{rank}(I_1 \oplus 0) = 1$.

Finally it is easy to prove that B is a Markov matrix. So

$$B = \begin{bmatrix} t_1X^t \\ \hline \vdots \\ \hline t_nX^t \end{bmatrix}$$

for some non–negative column vector X and where $t_1, \ldots, t_n$ are positive. We can assume that the entries of X sum to 1. It then follows that $t_1 = \cdots = t_n = 1$ and hence

$$B = \begin{bmatrix} X^t \\ \hline \vdots \\ \hline X^t \end{bmatrix}. \tag{19}$$

Now $A^m \to B$, so $A^{m+1} = A^m \cdot A \to BA$. Hence $B = BA$ and

$$A^t B^t = B^t. \tag{20}$$

Then equations (19) and (20) imply

$$A^t[X|\cdots|X] = [X|\cdots|X]$$

and hence $A^t X = X$.

However $X \geq 0$ and $A^t > 0$, so $X = A^t X > 0$.

DEFINITION 4.5

We have thus proved that there is a positive eigenvector X of A^t corresponding to the eigenvalue 1, where the components of X sum to 1. Then because we know that the eigenspace $N(A^t - I_n)$ is one–dimensional, it follows that this vector is unique.

This vector is called the **stationary vector** *of the Markov matrix A.*

EXAMPLE 4.4

Let

$$A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/6 & 1/6 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Then

$$A^t - I_3 \text{ row–reduces to } \begin{bmatrix} 1 & 0 & -4/9 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $N(A^t - I_3) = \left\langle \begin{bmatrix} 4/9 \\ 2/3 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 4/19 \\ 6/19 \\ 9/19 \end{bmatrix} \right\rangle$ *and*

$$\lim_{m\to\infty} A^m = \frac{1}{19}\begin{bmatrix} 4 & 6 & 9 \\ 4 & 6 & 9 \\ 4 & 6 & 9 \end{bmatrix}.$$

We remark that $ch_A = (x-1)(x^2 - 1/24)$.

DEFINITION 4.6

A Markov Matrix is called **regular** *or* **primitive** *if $\exists k \geq 1$ such that $A^k > 0$.*

THEOREM 4.12

If A is a primitive Markov matrix, then A satisfies the same properties enunciated in the last two theorems for positive Markov matrices.

PROOF Suppose $A^k > 0$. Then $(x-1)||\,\mathrm{ch}_{A^k}$ and hence $(x-1)||\,\mathrm{ch}_A$, as

$$\mathrm{ch}_A = (x-c_1)^{a_1}\cdots(x-c_t)^{a_t} \Rightarrow \mathrm{ch}_{A^k} = (x-c_1^k)^{a_1}\cdots(x-c_t^k)^{a_t}. \qquad (21)$$

and consequently $(x-1)||m_A$.

Also as 1 is the only eigenvalue of A^k with modulus 1, it follows from equation (21) that 1 is the only eigenvalue of A with modulus 1.

The proof of the second theorem goes through, with the difference that to prove the positivity of X we observe that $A^tX = X$ implies $(A^k)^tX = X$.

EXAMPLE 4.5

The following Markov matrix is primitive (its fourth power is positive) and is related to the $5x+1$ problem:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}.$$

Its stationary vector is $[\frac{1}{15}, \frac{2}{15}, \frac{8}{15}, \frac{4}{15}]^t$.

We remark that $ch_A = (x-1)(x+1/2)(x^2+1/4)$.