1 Linear Transformations

We will study mainly finite-dimensional vector spaces over an arbitrary field F—i.e. vector spaces with a basis. (Recall that the dimension of a vector space V ($\dim V$) is the number of elements in a basis of V.)

DEFINITION 1.1

(**Linear transformation**)

Given vector spaces U and V, $T : U \mapsto V$ is a linear transformation (LT) if

$$T(\lambda u + \mu v) = \lambda T(u) + \mu T(v)$$

for all $\lambda, \mu \in F$, and $u, v \in U$. Then $T(u+v) = T(u)+T(v)$, $T(\lambda u) = \lambda T(u)$ and

$$T\left(\sum_{k=1}^{n} \lambda_k u_k\right) = \sum_{k=1}^{n} \lambda_k T(u_k).$$

EXAMPLES 1.1

Consider the linear transformation

$$T = T_A : V_n(F) \mapsto V_m(F)$$

where $A = [a_{ij}]$ is $m \times n$, defined by $T_A(X) = AX$.

Note that $V_n(F)$ = the set of all n-dimensional column vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ of F—sometimes written F^n.

Note that if $T : V_n(F) \mapsto V_m(F)$ is a linear transformation, then $T = T_A$, where $A = [T(E_1)|\cdots|T(E_n)]$ and

$$E_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, E_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Note:

$$v \in V_n(F), \qquad v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 E_1 + \cdots + x_n E_n$$

If V is a vector space of all infinitely differentiable functions on $\mathbb{R}$, then

$$T(f) = a_0 \mathrm{D}^n f + a_1 \mathrm{D}^{n-1} f + \cdots + a_{n-1} \mathrm{D} f + a_n f$$

defines a linear transformation $T : V \mapsto V$.

The set of f such that $T(f) = 0$ (i.e. the kernel of T) is important.

Let $T : U \mapsto V$ be a linear transformation. Then we have the following definition:

DEFINITIONS 1.1

(**Kernel of a linear transformation**)

$$\operatorname{Ker} T = \{u \in U \mid T(u) = 0\}$$

(**Image of** T)

$$\operatorname{Im} T = \{v \in V \mid \exists u \in U \text{ such that } T(u) = v\}$$

Note: $\operatorname{Ker} T$ is a subspace of U. Recall that W is a subspace of U if

1. $0 \in W$,
2. W is closed under addition, and
3. W is closed under scalar multiplication.

PROOF. that $\operatorname{Ker} T$ is a subspace of U:

1. $T(0) + 0 = T(0) = T(0 + 0) = T(0) + T(0)$. Thus $T(0) = 0$, so $0 \in \operatorname{Ker} T$.
2. Let $u, v \in \operatorname{Ker} T$; then $T(u) = 0$ and $T(v) = 0$. So $T(u + v) = T(u) + T(v) = 0 + 0 = 0$ and $u + v \in \operatorname{Ker} T$.
3. Let $u \in \operatorname{Ker} T$ and $\lambda \in F$. Then $T(\lambda u) = \lambda T(u) = \lambda 0 = 0$. So $\lambda u \in \operatorname{Ker} T$.

EXAMPLE 1.1

$$\begin{aligned}
\operatorname{Ker} T_A &= N(A), \textit{ the null space of } A \\
&= \{X \in V_n(F) \mid AX = 0\} \\
\textit{and } \operatorname{Im} T_A &= C(A), \textit{ the column space of } A \\
&= \langle A_{*1}, \ldots, A_{*n} \rangle
\end{aligned}$$

Generally, if $U = \langle u_1, \ldots, u_n \rangle$, then $\operatorname{Im} T = \langle T(u_1), \ldots, T(u_n) \rangle$.

Note: Even if $u_1, \ldots, u_n$ form a basis for U, $T(u_1), \ldots, T(u_n)$ may not form a basis for $\operatorname{Im} T$. I.e. it may happen that $T(u_1), \ldots, T(u_n)$ are linearly dependent.

1.1 Rank + Nullity Theorems (for Linear Maps)

THEOREM 1.1 (General rank + nullity theorem)

If $T : U \mapsto V$ is a linear transformation then

$$\operatorname{rank} T + \operatorname{nullity} T = \dim U.$$

PROOF.

1. $\operatorname{Ker} T = \{0\}$.
   Then $\operatorname{nullity} T = 0$.
   We first show that the vectors $T(u_1), \ldots, T(u_n)$, where $u_1, \ldots, u_n$ are a basis for U, are LI (linearly independent):
   Suppose $x_1 T(u_1) + \cdots + x_n T(u_n) = 0$ where $x_1, \ldots, x_n \in F$.
   Then

$$\begin{aligned} T(x_1 u_1 + \cdots + x_n u_n) &= 0 \qquad \text{(by linearity)} \\ x_1 u_1 + \cdots + x_n u_n &= 0 \qquad \text{(since } \operatorname{Ker} T = \{0\}) \\ x_1 = 0, \ldots, x_n = 0 & \qquad \text{(since } u_i \text{ are LI)} \end{aligned}$$

   Hence $\operatorname{Im} T = \langle T(u_1), \ldots, T(u_n) \rangle$ so

$$\operatorname{rank} T + \operatorname{nullity} T = \dim \operatorname{Im} T + 0 = n = \dim V.$$

2. $\operatorname{Ker} T = U$.
   So $\operatorname{nullity} T = \dim U$.
   Hence $\operatorname{Im} T = \{0\} \Rightarrow \operatorname{rank} T = 0$

$$\begin{aligned} \Rightarrow \operatorname{rank} T + \operatorname{nullity} T &= 0 + \dim U \\ &= \dim U. \end{aligned}$$

3. $0 < \operatorname{nullity} T < \dim U$.
   Let $u_1, \ldots, u_r$ be a basis for $\operatorname{Ker} T$ and $n = \dim U$, so $r = \operatorname{nullity} T$ and $r < n$.
   Extend the basis $u_1, \ldots, u_r$ to form a basis $u_1, \ldots, u_r, u_{r+1}, \ldots, u_n$ of

U (refer to last year's notes to show that this can be done).
Then $T(u_{r+1}), \ldots, T(u_n)$ span $\operatorname{Im} T$. For

$$\begin{aligned}
\operatorname{Im} T &= \langle T(u_1), \ldots, T(u_r), T(u_{r+1}), \ldots, T(u_n) \rangle \\
&= \langle 0, \ldots, 0, T(u_{r+1}), \ldots, T(u_n) \rangle \\
&= \langle T(u_{r+1}), \ldots, T(u_n) \rangle
\end{aligned}$$

So assume

$$\begin{aligned}
& x_1 T(u_{r+1}) + \cdots + x_{n-r} T(u_n) = 0 \\
\Rightarrow\ & T(x_1 u_{r+1} + \cdots + x_{n-r} u_n) = 0 \\
\Rightarrow\ & x_1 u_{r+1} + \cdots + x_{n-r} u_n \in \operatorname{Ker} T \\
\Rightarrow\ & x_1 u_{r+1} + \cdots + x_{n-r} u_n = y_1 u_1 + \cdots + y_r u_r \\
& \qquad \text{for some } y_1, \ldots, y_r \\
\Rightarrow\ & (-y_1) u_1 + \cdots + (-y_r) u_r + x_1 u_{r+1} + \cdots + x_{n-r} u_n = 0
\end{aligned}$$

and since $u_1, \ldots, u_n$ is a basis for U, all coefficients vanish.
Thus

$$\begin{aligned}
\operatorname{rank} T + \operatorname{nullity} T &= (n-r) + r \\
&= n \\
&= \dim U.
\end{aligned}$$

We now apply this theorem to prove the following result:

THEOREM 1.2 (Dimension theorem for subspaces)

$$\dim(U \cap V) + \dim(U + V) = \dim U + \dim V$$

where U and V are subspaces of a vector space W.

(Recall that $U + V = \{u + v \mid u \in U, v \in V\}$.)
For the proof we need the following definition:

DEFINITION 1.2
If U and V are any two vector spaces, then the direct sum is

$$U \oplus V = \{(u, v) \mid u \in U, v \in V\}$$

(i.e. the cartesian product of U and V) made into a vector space by the component-wise definitions:

1. $(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$,
2. $\lambda(u, v) = (\lambda u, \lambda v)$, *and*
3. $(0, 0)$ *is an identity for* $U \oplus V$ *and* $(-u, -v)$ *is an additive inverse for* (u, v).

We need the following result:

THEOREM 1.3

$$\dim(U \oplus V) = \dim U + \dim V$$

PROOF.

Case 1: $U = \{0\}$

Case 2: $V = \{0\}$
Proof of cases 1 and 2 are left as an exercise.

Case 3: $U \neq \{0\}$ and $V \neq \{0\}$

Let $u_1, \ldots, u_m$ be a basis for U, and
$v_1, \ldots, v_n$ be a basis for V.

We assert that $(u_1, 0), \ldots, (u_m, 0), (0, v_1), \ldots, (0, v_n)$ form a basis for $U \oplus V$.
Firstly, spanning:
Let $(u, v) \in U \oplus V$, say $u = x_1 u_1 + \cdots + x_m u_m$ and $v = y_1 v_1 + \cdots + y_n v_n$. Then

$$\begin{aligned}
(u, v) &= (u, 0) + (0, v) \\
&= (x_1 u_1 + \cdots + x_m u_m, 0) + (0, y_1 v_1 + \cdots + y_n v_n) \\
&= x_1(u_1, 0) + \cdots + x_m(u_m, 0) + y_1(0, v_1) + \cdots + y_n(0, v_n) \\
\text{So} \quad U \oplus V &= \langle (u_1, 0), \ldots, (u_m, 0), (0, v_1), \ldots, (0, v_n) \rangle
\end{aligned}$$

Secondly, independence: assume $x_1(u_1, 0) + \cdots + x_m(u_m, 0) + y_1(0, v_1) + \cdots + y_n(0, v_n) = (0, 0)$. Then

$$\begin{aligned}
(x_1 u_1 + \cdots + x_m u_m, y_1 v_1 + \cdots + y_n v_n) &= 0 \\
\Rightarrow \quad x_1 u_1 + \cdots + x_m u_m &= 0 \\
\text{and } y_1 v_1 + \cdots + y_n v_n &= 0 \\
\Rightarrow \quad x_i &= 0, \ \forall i \\
\text{and } y_i &= 0, \ \forall i
\end{aligned}$$

Hence the assertion is true and the result follows.

PROOF.

Let $T : U \oplus V \mapsto U + V$ where U and V are subspaces of some W, such that $T(u, v) = u + v$.

Thus $\operatorname{Im} T = U + V$, and

$$\begin{aligned} \operatorname{Ker} T &= \{(u, v) \mid u \in U, v \in V, \text{ and } u + v = 0\} \\ &= \{(t, -t) \mid t \in U \cap V\} \end{aligned}$$

Clearly then, dim $\operatorname{Ker} T = \dim(U \cap V)$[1] and so

$$\begin{aligned} \operatorname{rank} T + \operatorname{nullity} T &= \dim(U \oplus V) \\ \Rightarrow \quad \dim(U + V) + \dim(U \cap V) &= \dim U + \dim V. \end{aligned}$$

1.2 Matrix of a Linear Transformation

DEFINITION 1.3

Let $T : U \mapsto V$ be a LT with bases $\beta : u_1, \ldots, u_n$ and $\gamma : v_1, \ldots, v_m$ for U and V respectively.

Then

$$T(u_j) = \begin{array}{c} a_{1j}v_1 \\ + \\ a_{2j}v_2 \\ + \\ \vdots \\ + \\ a_{mj}v_m \end{array} \quad \textit{for some} \quad \begin{array}{c} a_{1j} \\ \vdots \\ a_{mj} \end{array} \in F.$$

The $m \times n$ matrix

$$A = [a_{ij}]$$

is called the matrix of T relative to the bases β and γ and is also written

$$A = [T]_\beta^\gamma$$

Note: *The j-th column of A is the co-ordinate vector of $T(u_j)$, where u_j is the j-th vector of the basis β.*

Also if $u = x_1u_1 + \cdots + x_nu_n$, the co-ordinate vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is denoted by $[u]_\beta$.

[1] True if $U \cap V = \{0\}$; if not, let $S = \operatorname{Ker} T$ and $u_1, \ldots, u_r$ be a basis for $U \cap V$. Then $(u_1, -u_1), \ldots, (u_r, -u_r)$ form a basis for S and hence dim $\operatorname{Ker} T = \dim S$.

EXAMPLE 1.2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(F)$ *and let* $T : M_{2\times 2}(F) \mapsto M_{2\times 2}(F)$ *be defined by*

$$T(X) = AX - XA.$$

Then T *is linear*[2], *and* $\operatorname{Ker} T$ *consists of all* 2×2 *matrices* A *where* $AX = XA$.

Take β *to be the basis* E_{11}, E_{12}, E_{21}, *and* E_{22}, *defined by*

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(so we can define a matrix for the transformation, consider these henceforth to be column vectors of four elements).

Calculate $[T]_\beta^\beta = B$:

$$\begin{aligned} T(E_{11}) &= AE_{11} - E_{11}A \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} \\ &= 0E_{11} - bE_{12} + cE_{21} + 0E_{22} \end{aligned}$$

and similar calculations for the image of other basis vectors show that

$$B = \begin{bmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & b & 0 \end{bmatrix}$$

EXERCISE: *Prove that* $\operatorname{rank} B = 2$ *if* A *is not a scalar matrix (i.e. if* $A \neq tI_n$*).*

Later, we will show that $\operatorname{rank} B = \operatorname{rank} T$. *Hence*

$$\operatorname{nullity} T = 4 - 2 = 2$$

[2]

$$\begin{aligned} T(\lambda X + \mu Y) &= A(\lambda X + \mu Y) - (\lambda X + \mu Y)A \\ &= \lambda(AX - XA) + \mu(AY - YA) \\ &= \lambda T(X) + \mu T(Y) \end{aligned}$$

Note: *$I_2, A \in \operatorname{Ker} T$ which has dimension 2. Hence if A is not a scalar matrix, since I_2 and A are LI they form a basis for $\operatorname{Ker} T$. Hence*

$$AX = XA \quad \Rightarrow \quad X = \alpha I_2 + \beta A.$$

DEFINITIONS 1.2

Let T_1 and T_2 be LT's mapping U to V.
Then $T_1 + T_2 : U \mapsto V$ is defined by

$$(T_1 + T_2)(x) = T_1(x) + T_2(x) \quad ; \forall x \in U$$

For T a LT and $\lambda \in F$, define $\lambda T : U \mapsto V$ by

$$(\lambda T)(x) = \lambda T(x) \ \forall x \in U$$

Now ...

$$\begin{aligned} [T_1 + T_2]_\beta^\gamma &= [T_1]_\beta^\gamma + [T_2]_\beta^\gamma \\ [\lambda T]_\beta^\gamma &= \lambda [T]_\beta^\gamma \end{aligned}$$

DEFINITION 1.4

$$\operatorname{Hom}(U, V) = \{T | T : U \mapsto V \text{ is a LT}\}.$$

$\operatorname{Hom}(U, V)$ *is sometimes written $L(U, V)$.*

The zero transformation $0 : U \mapsto V$ is such that $0(x) = 0$, $\forall x$.
If $T \in \operatorname{Hom}(U, V)$, then $(-T) \in \operatorname{Hom}(U, V)$ is defined by

$$(-T)(x) = -(T(x)) \quad \forall x \in U.$$

Clearly, $\operatorname{Hom}(U, V)$ is a vector space.
Also

$$\begin{aligned} [0]_\beta^\gamma &= 0 \\ \text{and} \quad [-T]_\beta^\gamma &= -[T]_\beta^\gamma \end{aligned}$$

The following result reduces the computation of $T(u)$ to matrix multiplication:

THEOREM 1.4

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

PROOF.

Let $A = [T]_\beta^\gamma$, where β is the basis $u_1, \ldots, u_n$, γ is the basis $v_1, \ldots, v_m$, and

$$T(u_j) = \sum_{i=1}^{m} a_{ij} v_i.$$

Also let $[u]_\beta = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Then $u = \sum_{j=1}^{n} x_j u_j$, so

$$\begin{aligned} T(u) &= \sum_{j=1}^{n} x_j T(u_j) \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij} v_i \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) v_i \\ \Rightarrow [T(u)]_\gamma &= \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} \\ &= A[u]_\beta \end{aligned}$$

DEFINITION 1.5

*(**Composition** of LTs)*

If $T_1 : U \mapsto V$ and $T_2 : V \mapsto W$ are LTs, then $T_2T_1 : U \mapsto W$ defined by

$$(T_2T_1)(x) = T_2(T_1(x)) \qquad \forall x \in U$$

is a LT.

THEOREM 1.5

If β, γ and δ are bases for U, V and W, then

$$[T_2T_1]_\beta^\delta = [T_2]_\gamma^\delta [T_1]_\beta^\gamma$$

PROOF. Let $u \in U$. Then

$$\begin{aligned} [T_2T_1(u)]_\delta &= [T_2T_1]_\beta^\delta [u]_\beta \\ \text{and} &= [T_2(T_1(u))]_\delta \\ &= [T_2]_\gamma^\delta [T_1(u)]_\gamma \end{aligned}$$

Hence

$$[T_2T_1]_\beta^\delta [u]_\beta = [T_2]_\gamma^\delta [T_1]_\beta^\gamma [u]_\beta \qquad (1)$$

(note that we can't just "cancel off" the $[u]_\beta$ to obtain the desired result!)

Finally, if β is $u_1, \ldots, u_n$, note that $[u_j]_\beta = E_j$ (since $u_j = 0u_1 + \cdots + 0u_{j-1} + 1u_j + 0u_{j+1} + \cdots + 0u_n$) then for an appropriately sized matrix B,

$$BE_j = B_{*j}, \quad \text{the } j\text{th column of } B.$$

Then (1) shows that the matrices

$$[T_2T_1]_\beta^\delta \quad \text{and} \quad [T_2]_\gamma^\delta [T_1]_\beta^\gamma$$

have their first, second, ..., nth columns respectively equal.

EXAMPLE 1.3

If A is $m \times n$ and B is $n \times p$, then

$$T_A T_B = T_{AB}.$$

DEFINITION 1.6

(the **identity transformation***)*

Let U be a vector space. Then the identity transformation $I_U : U \mapsto U$ defined by

$$I_U(x) = x \quad \forall x \in U$$

is a linear transformation, and

$$[I_U]_\beta^\beta = I_n \quad \textit{if } n = \dim U.$$

Also note that $I_{V_n(F)} = T_{I_n}$.

THEOREM 1.6

Let $T : U \mapsto V$ be a LT. Then

$$I_V T = T I_U = T.$$

Then

$$T_{I_m}T_A = T_{I_mA} = T_A = T_AT_{AI_n} = T_{AI_n}$$

and consequently we have the familiar result

$$I_mA = A = AI_n.$$

DEFINITION 1.7

(**Invertible LTs**)

Let $T : U \mapsto V$ be a LT.

If $\exists S : V \mapsto U$ such that S is linear and satisfies

$$ST = I_U \quad \text{and} \quad TS = I_V$$

then we say that T is **invertible** *and that S is an* **inverse** *of T.*

Such inverses are unique and we thus denote S by T^{-1}.

Explicitly,

$$S(T(x)) = x \; \forall x \in U \quad \text{and} \quad T(S(y)) = y \; \forall y \in V$$

There is a corresponding definition of an **invertible matrix***: $A \in M_{m\times n}(F)$ is called invertible if $\exists B \in M_{n\times m}(F)$ such that*

$$AB = I_m \quad \text{and} \quad BA = I_n$$

Evidently

THEOREM 1.7

T_A is invertible iff A is invertible (i.e. if A^{-1} exists). Then,

$$(T_A)^{-1} = T_{A^{-1}}$$

THEOREM 1.8

If $u_1, \ldots, u_n$ is a basis for U and $v_1, \ldots, v_n$ are vectors in V, then there is one and only one linear transformation $T : U \to V$ satisfying

$$T(u_1) = v_1, \ldots, T(u_n) = v_n,$$

namely $T(x_1u_1 + \cdots + x_nu_n) = x_1v_1 + \cdots + x_nv_n$.

(In words, a linear transformation is determined by its action on a basis.)

1.3 Isomorphisms

DEFINITION 1.8

A linear map $T : U \mapsto V$ is called an **isomorphism** *if T is 1-1 and onto, i.e.*

1. $T(x) = T(y) \Rightarrow x = y \; \forall x, y \in U$, *and*
2. $\operatorname{Im} T = V$, *that is, if $v \in V$, $\exists u \in U$ such that $T(u) = v$.*

Lemma: A linear map T is 1-1 iff $\operatorname{Ker} T = \{0\}$.

PROOF:

1. ($\Rightarrow$) Suppose T is 1-1 and let $x \in \operatorname{Ker} T$.
   We have $T(x) = 0 = T(0)$, and so $x = 0$.
2. ($\Leftarrow$) Assume $\operatorname{Ker} T = \{0\}$ and $T(x) = T(y)$ for some $x, y \in U$.
   Then

$$\begin{aligned} T(x-y) &= T(x) - T(y) = 0 \\ \Rightarrow x - y &\in \operatorname{Ker} T \\ \Rightarrow x - y &= 0 \Rightarrow x = y \end{aligned}$$

THEOREM 1.9

Let $A \in M_{m \times n}(F)$. Then $T_A : V_n(F) \to V_m(F)$ is

(a) *onto:* $\Leftrightarrow \dim C(A) = m \Leftrightarrow$ *the rows of A are LI;*

(b) *1–1:* $\Leftrightarrow \dim N(A) = 0 \Leftrightarrow \operatorname{rank} A = n \Leftrightarrow$ *the columns of A are LI.*

EXAMPLE 1.4

Let $T_A : V_n(F) \mapsto V_n(F)$ with A invertible; so $T_A(X) = AX$.
We will show this to be an isomorphism.

1. *Let $X \in \operatorname{Ker} T_A$, i.e. $AX = 0$. Then*

$$\begin{aligned} A^{-1}(AX) &= A^{-1}0 \\ \Rightarrow I_n X &= 0 \\ \Rightarrow X &= 0 \\ \Rightarrow \operatorname{Ker} T &= \{0\} \\ &\Leftrightarrow T \text{ is 1-1.} \end{aligned}$$

2. Let $Y \in V_n(F)$: then,

$$\begin{aligned} T(A^{-1}Y) &= A(A^{-1}Y) \\ &= I_nY = Y \\ \text{so} \quad \operatorname{Im} T_A &= V_n(F) \end{aligned}$$

THEOREM 1.10

If T is an isomorphism between U and V, then

$$\dim U = \dim V$$

PROOF.

Let $u_1, \ldots, u_n$ be a basis for U. Then

$$T(u_1), \ldots, T(u_n)$$

is a basis for V (i.e. $\langle u_i \rangle = U$ and $\langle T(u_i) \rangle = V$, with u_i, v_i independent families), so

$$\dim U = n = \dim V$$

THEOREM 1.11

$$\Phi : \operatorname{Hom}(U, V) \mapsto M_{m\times n}(F) \quad \textit{defined by} \quad \Phi(T) = [T]_\beta^\gamma$$

is an isomorphism.

Here $\dim U = n$, $\dim V = m$, and β and γ are bases for U and V, respectively.

THEOREM 1.12

$$\begin{aligned} & T : U \mapsto V \textit{ is invertible} \\ \Leftrightarrow \quad & T \textit{ is an isomorphism between } U \textit{ and } V. \end{aligned}$$

PROOF.

$\Rightarrow$ Assume T is invertible. Then

$$\begin{aligned} T^{-1}T &= I_U \\ \text{and} \quad TT^{-1} &= I_V \\ \Rightarrow T^{-1}(T(x)) &= x \ \forall x \in U \\ \text{and } T(T^{-1}(y)) &= y \ \forall y \in V \end{aligned}$$

1. We prove $\operatorname{Ker} T = \{0\}$.
   Let $T(x) = 0$. Then
$$T^{-1}(T(x)) = T^{-1}(0) = 0 = x$$
   So T is 1-1.
2. We show $\operatorname{Im} T = V$.
   Let $y \in V$. Now $T(T^{-1}(y)) = y$, so taking $x = T^{-1}(y)$ gives
$$T(x) = y.$$
   Hence $\operatorname{Im} T = V$.

$\Leftarrow$ Assume T is an isomorphism, and let S be the inverse map of T
$$S : V \mapsto U$$
Then $ST = I_U$ and $TS = I_V$. It remains to show that S is linear.

We note that
$$x = S(y) \Leftrightarrow y = T(x)$$
And thus, using linearity of T only, for any $y_1, y_2 \in V$, $x_1 = S(y_1)$, and $x_2 = S(y_2)$ we obtain
$$\begin{aligned} S(\lambda y_1 + \mu y_2) &= S(\lambda T(x_1) + \mu T(x_2)) \\ &= S(T(\lambda x_1 + \mu x_2)) \\ &= \lambda x_1 + \mu x_2 \\ &= \lambda S(y_1) + \mu S(y_2) \end{aligned}$$

COROLLARY 1.1
If $A \in M_{m\times n}(F)$ is invertible, then $m = n$.

PROOF.
Suppose A is invertible. Then T_A is invertible and thus an isomorphism between $V_n(F)$ and $V_m(F)$.
Hence $\dim V_n(F) = \dim V_m(F)$ and hence $m = n$.

THEOREM 1.13
If $\dim U = \dim V$ and $T : U \mapsto V$ is a LT, then
$$\begin{aligned} T \text{ is 1-1 (injective)} &\Leftrightarrow T \text{ is onto (surjective)} \\ (&\Leftrightarrow T \text{ is an isomorphism} \quad) \end{aligned}$$

PROOF.

$\Rightarrow$ Suppose T is 1-1.

Then $\operatorname{Ker} T = \{0\}$ and we have to show that $\operatorname{Im} T = V$.

$$\begin{aligned} \operatorname{rank} T + \operatorname{nullity} T &= \dim U \\ \Rightarrow \operatorname{rank} T + 0 &= \dim V \\ \text{i.e. } \dim(\operatorname{Im} T) &= \dim V \\ \Rightarrow \operatorname{Im} T &= V \quad \text{as } T \subseteq V. \end{aligned}$$

$\Leftarrow$ Suppose T is onto.

Then $\operatorname{Im} T = V$ and we must show that $\operatorname{Ker} T = \{0\}$. The above argument is reversible:

$$\begin{aligned} \operatorname{Im} T &= V \\ \operatorname{rank} T &= \dim V \\ &= \dim U \\ &= \operatorname{rank} T + \operatorname{nullity} T \\ \Rightarrow \operatorname{nullity} T &= 0 \\ \text{or} \quad \operatorname{Ker} T &= \{0\} \end{aligned}$$

COROLLARY 1.2

Let $A, B \in M_{n\times n}(F)$. *Then*

$$AB = I_n \Rightarrow BA = I_n.$$

PROOF Suppose $AB = I_n$. Then $\operatorname{Ker} T_B = \{0\}$. For

$$\begin{aligned} BX = 0 &\Rightarrow A(BX) = A0 = 0 \\ &\Rightarrow I_n X = 0 \Rightarrow X = 0. \end{aligned}$$

But $\dim U = \dim V = n$, so T_B is an isomorphism and hence invertible.

Thus $\exists C \in M_{n\times n}(F)$ such that

$$\begin{aligned} T_B T_C &= I_{V_n(F)} = T_C T_B \\ \Rightarrow BC &= I_n = CB, \end{aligned}$$

noting that $I_{V_n(F)} = T_{I_n}$.

Now, knowing $AB = I_n$,

$$\begin{aligned} \Rightarrow A(BC) &= A \\ (AB)C &= A \\ I_nC &= A \\ \Rightarrow C &= A \\ \Rightarrow BA &= I_n \end{aligned}$$

DEFINITION 1.9

Another standard isomorphism: Let $\dim V = m$*, with basis* $\gamma = v_1, \ldots, v_m$*. Then* $\phi_\gamma : V \mapsto V_m(F)$ *is the isomorphism defined by*

$$\phi_\gamma(v) = [v]_\gamma$$

THEOREM 1.14

$$\operatorname{rank} T = \operatorname{rank} [T]_\beta^\gamma$$

PROOF

$$\begin{array}{ccc} U & \xrightarrow{T} & V \\ \phi_\beta \quad \downarrow & & \downarrow \quad \phi_\gamma \\ V_n(F) & \overrightarrow{T_A} & V_m(F) \end{array}$$

With

$$\begin{array}{l} \beta : u_1, \ldots, u_n \\ \gamma : v_1, \ldots, v_m \end{array} \text{ a basis for } \begin{array}{l} U \\ V, \end{array}$$

let $A = [T]_\beta^\gamma$. Then the commutative diagram is an abbreviation for the equation

$$\phi_\gamma T = T_A \phi_\beta. \qquad (2)$$

Equivalently

$$\phi_\gamma T(u) = T_A \phi_\beta(u) \ \forall u \in U$$

or

$$[T(u)]_\gamma = A[u]_\beta$$

which we saw in Theorem 1.4.

But $\operatorname{rank}(ST) = \operatorname{rank} T$ if S is invertible and $\operatorname{rank}(TR) = \operatorname{rank} T$ if R is invertible. Hence, since ϕ_β and ϕ_γ are both invertible,

$$(2) \ \Rightarrow \ \operatorname{rank} T = \operatorname{rank} T_A = \operatorname{rank} A$$

and the result is proven.

Note:

Observe that $\phi_\gamma(T(u_j)) = A_{*j}$, the jth column of A. So $\operatorname{Im} T$ is mapped under ϕ_γ into $C(A)$. Also $\operatorname{Ker} T$ is mapped by ϕ_β into $N(A)$. Consequently we get bases for $\operatorname{Im} T$ and $\operatorname{Ker} T$ from bases for $C(A)$ and $N(A)$, respectively.

$$\begin{aligned}(u \in \operatorname{Ker} T \Leftrightarrow T(u) = 0 \quad &\Leftrightarrow \quad \phi_\gamma(T(u)) = 0 \\ &\Leftrightarrow \quad T_A \phi_\beta(u) = 0 \\ &\Leftrightarrow \quad \phi_\beta(u) \in N(A).)\end{aligned}$$

THEOREM 1.15

Let β and γ be bases for some vector space V. Then, with $n = \dim V$,

$$[I_V]_\beta^\gamma$$

is non-singular and its inverse

$$\left\{[I_V]_\beta^\gamma\right\}^{-1} = [I_V]_\gamma^\beta.$$

PROOF

$$\begin{aligned} I_V I_V &= I_V \\ \Rightarrow [I_V I_V]_\beta^\beta &= [I_V]_\beta^\beta = I_n \\ &= [I_V]_\gamma^\beta [I_V]_\beta^\gamma. \end{aligned}$$

The matrix $P = [I_V]_\beta^\gamma = [p_{ij}]$ is called the *change of basis matrix.* For if $\beta : u_1, \ldots, u_n$ and $\gamma : v_1, \ldots, v_n$ then

$$\begin{aligned} u_j &= I_V(u_j) \\ &= p_{1j}v_1 + \cdots + p_{nj}v_n \quad \text{for } j = 1, \ldots, n. \end{aligned}$$

It is also called the *change of co-ordinate matrix*, since

$$[v]_\gamma = [I_V(v)]_\beta^\gamma [v]_\beta$$

i.e. if

$$\begin{aligned} v &= x_1u_1 + \cdots + x_nu_n \\ &= y_1v_1 + \cdots + y_nv_n \end{aligned}$$

then

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = P \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

or, more explicitly,

$$\begin{aligned} y_1 &= p_{11}x_1 + \cdots + p_{1n}x_n \\ &\vdots \\ y_n &= p_{n1}x_1 + \cdots + p_{1n}x_n. \end{aligned}$$

THEOREM 1.16 (Effect of changing basis on matrices of LTs)

Let $T : V \mapsto V$ be a LT with bases β and γ. Then

$$[T]_\beta^\beta = P^{-1}[T]_\gamma^\gamma P$$

where

$$P = [I_V]_\beta^\gamma$$

as above.

PROOF

$$\begin{aligned} I_V T &= T = T I_V \\ \Rightarrow [I_V T]_\beta^\gamma &= [T I_V]_\beta^\gamma \\ \Rightarrow [I_V]_\beta^\gamma [T]_\beta^\beta &= [T]_\gamma^\gamma [I_V]_\beta^\gamma \end{aligned}$$

DEFINITION 1.10

*(**Similar matrices**)*

If A and B are two matrices in $M_{m\times n}(F)$, then if there exists a non-singular matrix P such that

$$B = P^{-1}AP$$

we say that A and B are **similar** *over F.*

1.4 Change of Basis Theorem for T_A

In the MP274 course we are often proving results about linear transformations $T : V \mapsto V$ which state that a basis β can be found for V so that $[T]_\beta^\beta = B$, where B has some special property. If we apply the result to the linear transformation $T_A : V_n(F) \mapsto V_n(F)$, the change of basis theorem applied to T_A tells us that A is similar to B. More explicitly, we have the following:

THEOREM 1.17

Let $A \in M_{n\times n}(F)$ and suppose that $v_1, \ldots, v_n \in V_n(F)$ form a basis β for $V_n(F)$. Then if $P = [v_1|\cdots|v_n]$ we have

$$P^{-1}AP = [T_A]_\beta^\beta.$$

PROOF. Let γ be the standard basis for $V_n(F)$ consisting of the unit vectors $E_1, \ldots, E_n$ and let $\beta : v_1, \ldots, v_n$ be a basis for $V_n(F)$. Then the change of basis theorem applied to $T = T_A$ gives

$$[T_A]_\beta^\beta = P^{-1}[T_A]_\gamma^\gamma P,$$

where $P = [I_V]_\beta^\gamma$ is the change of coordinate matrix.

Now the definition of P gives

$$\begin{aligned} v_1 = I_V(v_1) &= p_{11}E_1 + \cdots + p_{n1}E_n \\ &\vdots \\ v_n = I_V(v_n) &= p_{1n}E_1 + \cdots + p_{nn}E_n, \end{aligned}$$

or, more explicitly,

$$v_1 = \begin{bmatrix} p_{11} \\ \vdots \\ p_{n1} \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} p_{1n} \\ \vdots \\ p_{nn} \end{bmatrix}.$$

In other words, $P = [v_1|\cdots|v_n]$, the matrix whose columns are $v_1, \ldots, v_n$ respectively.

Finally, we observe that $[T_A]_\gamma^\gamma = A$.