4.2 Two Jordan Canonical Form Examples

4.2.1 Example (a):

Let \( A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix} \in M_{4 \times 4}(\mathbb{Q}). \)

We find \( \text{ch}_A = (x - 2)^2(x - 3)^2 = p_1^2 p_2^2 \), where \( p_1 = x - 2, \ p_2 = x - 3. \)

**CASE 1, \( p_1 = x - 2: \)**

\[
p_1(A) = A - 2I_4 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

so \( \nu(p_1(A)) = \gamma_1 = 2. \) Hence \( b_1 = 1 \) and the corresponding dot diagram has height 1, width 2, with associated Jordan blocks \( J_1(2) \oplus J_1(2): \)

\[ \begin{array}{ccc} \\
\end{array} \] \( N_{1,x-2} \)

We find \( v_{11} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \) and \( v_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \) form a basis for \( \text{Ker}p_1(T_A) = N(A - 2I_4) \) and \( m_{T_A,v_{11}} = m_{T_A,v_{12}} = x - 2. \) Also

\[
\text{Ker}(p_1^{b_1}(T_A)) = N(p_1(A)) = N(A - 2I_4) = C_{T_A,v_{11}} \oplus C_{T_A,v_{12}}.
\]

Note that \( C_{T_A,v_{11}} \) and \( C_{T_A,v_{12}} \) have Jordan bases \( \beta_{11} : v_{11} \) and \( \beta_{12} : v_{12} \) respectively.

**CASE 2, \( p_2 = x - 3: \)**

\[
p_2(A) = A - 3I_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 4 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

so \( \nu(p_2(A)) = 1 = \gamma_2; \) also \( \nu(p_2^2(A)) = 2. \) Hence \( b_2 = 2 \) and we get a corresponding dot diagram consisting of two vertical dots, with associated Jordan block \( J_2(3): \)

\[ \begin{array}{ccc} \\
\end{array} \] \( N_{2,x-3} \)

\[ \begin{array}{ccc} \\
\end{array} \] \( N_{1,x-3} \)

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We have to find a basis of the form $p_2(T_A)(v_{21}) = (A-3I_4)v_{21}$ for $\text{Ker} p_2(T_A) = N(A-3I_4)$.

To find $v_{21}$ we first get a basis for $N(A-3I_4)^2$. We have

$$p_2^2(A) = (A-3I_4)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -10 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we find $X_1 = \begin{bmatrix} 2 \\ 10 \\ 1 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ is such a basis. Then we have

$$N_{2,p_2} = \langle p_2X_1, p_2X_2 \rangle = \langle p_2(A)X_1, p_2(A)X_2 \rangle = \langle (A-3I_4)X_1, (A-3I_4)X_2 \rangle$$

$$= \begin{bmatrix} 3 \\ -3 \\ -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \\ 9 \end{bmatrix}.$$ 

Hence we can take $v_{21} = X_1$. Then $m_{T_A,v_{21}} = (x-3)^2$. Also

$$\text{Ker} p_2^{b_2}(T_A)) = N(p_2^2(A)) = N(A-3I_4)^2 = C_{T_A,v_{21}}.$$ 

Moreover $C_{T_A,v_{21}}$ has Jordan basis $\beta_{21} : v_{21}, (A-3I_4)v_{21}$.

Finally we have $V_4(Q) = C_{T_{A,v_{11}}} \oplus C_{T_{A,v_{12}}} \oplus C_{T_{A,v_{21}}}$ and $\beta = \beta_{11} \cup \beta_{12} \cup \beta_{21}$ is a basis for $V_4(Q)$. Then with

$$P = [v_{11} | v_{12} | v_{21} | (A-3I_4)v_{21}] = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 1 & 0 & 10 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 9 \end{bmatrix}$$

we have

$$P^{-1}AP = [T_A]_{\beta}^\beta = J_1(2) \oplus J_1(2) \oplus J_2(3) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$
4.2.2 Example (b):

Let $A \in M_{6 \times 6}(F)$ have the property that $ch_A = x^6$, $m_A = x^3$ and

$$\nu(A) = 3, \nu(A^2) = 5, (\nu(A^3) = 6).$$

Next, with $\nu_{h,x} = \dim_F N_{h,x}$ we have

$$\nu_{1,x} = \nu(A) = 3 = \gamma;$$
$$\nu_{2,x} = \nu(A^2) - \nu(A) = 5 - 3 = 2;$$
$$\nu_{3,x} = \nu(A^3) - \nu(A^2) = 6 - 5 = 1.$$

Hence the dot diagram corresponding to the (only) monic irreducible factor $x$ of $m_A$ is

```
•     •     •
•     •     •
•     •     •
N_3,x
N_2,x
N_1,x
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Hence we read off that $\exists$ a non-singular $P \in M_{6 \times 6}(F)$ such that $P^{-1}AP = J_3(0) \oplus J_2(0) \oplus J_1(0)$. To find such a matrix $P$ we proceed as follows:

(i) First find a basis for $N_{3,x}$. We do this by first finding a basis for $N(A^3): X_1, X_2, X_3, X_4, X_5, X_6$. Then

$$N_{3,x} = \langle A^2X_1, A^2X_2, A^2X_3, A^2X_4, A^2X_5, A^2X_6 \rangle.$$

We now apply the LRA (left-to-right algorithm) to the above spanning family to get a basis $A^2v_{11}$ for $N_{3,x}$, where $A^2v_{11}$ is the first non-zero vector in the spanning family.

(ii) Now extend the linearly independent family $A^2v_{11}$ to a basis for $N_{2,x}$. We do this by first finding a basis $Y_1, Y_2, Y_3, Y_4, Y_5$ for $N(A^2)$. Then

$$N_{2,x} = \langle AY_1, AY_2, AY_3, AY_4, AY_5 \rangle.$$

We now attach $A^2v_{11}$ to the head of this spanning family:

$$N_{2,x} = \langle A^2v_{11}, AY_1, AY_2, AY_3, AY_4, AY_5 \rangle$$

and apply the LRA to find a basis for $N_{2,x}$ which includes $A^2X_1$. This will have the form $A^2v_{11}, Av_{12}$, where $Av_{12}$ is the first vector in the list $AY_1, \ldots, AY_5$ which is not a linear combination of $A^2v_{11}$.

(iii) Now extend the linearly independent family $A^2v_{11}, Av_{12}$ to a basis for $N_{1,x} = N(A)$. We do this by first finding a basis $Z_1, Z_2, Z_3$ for $N(A)$.

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Then place the linearly independent family $A^2v_{11}, Av_{12}$ at the head of this spanning family:

$$N_{1,x} = (A^2v_{11}, Av_{12}, Z_1, Z_2, Z_3).$$

The LRA is then applied to the above spanning family selects a basis of the form $A^2v_{11}, Av_{12}, v_{13}$, where $v_{13}$ is the first vector among $Z_1, Z_2, Z_3$ which is not a linear combination of $A^2v_{11}$ and $Av_{12}$.

Then $m_{TA}, v_{11} = x^3$, $m_{TA}, v_{12} = x^2$, $m_{TA}, v_{13} = x$. Also

$$\text{Ker} \ p^h_1(T_A) = N(A^3) = C_{TA, v_{11}} \oplus C_{TA, v_{12}} \oplus C_{TA, v_{13}}.$$ 

Finally, if we take Jordan bases

$$\beta_{11} : v_{11}, Av_{11}, A^2v_{11};$$
$$\beta_{12} : v_{12}, Av_{12};$$
$$\beta_{13} : v_{13}$$

for the three $T$–cyclic subspaces $C_{TA, v_{11}}, C_{TA, v_{12}}, C_{TA, v_{13}}$, respectively, we then get the basis

$$\beta = \beta_{11} \cup \beta_{12} \cup \beta_{13} = v_{11}, Av_{11}, A^2v_{11}; v_{12}, Av_{12}; v_{13}$$

for $V_6(F)$. Then if

$$P = [v_{11}|Av_{11}|A^2v_{11}|v_{12}|Av_{12}|v_{13}]$$

we have

$$P^{-1}AP = [T_A]^{\beta} = J_3(0) \oplus J_2(0) \oplus J_1(0) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. $$