4.2 Two Jordan Canonical Form Examples

4.2.1 Example (a):

Let $A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix} \in M_{4\times 4}(\mathbb{Q})$.

We find $ch_T = (x-2)^2(x-3)^2 = p_1^2 p_2^2$, where $p_1 = x-2,\ p_2 = x-3$.

CASE 1, $p_1 = x - 2$:

$$p_1(A) = A - 2I_4 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \end{bmatrix} \quad\rightarrow\quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $\nu(p_1(A)) = \gamma_1 = 2$. Hence $b_1 = 1$ and the corresponding dot diagram has height 1, width 2, with associated Jordan blocks $J_1(2) \oplus J_1(2)$:

$$\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array} \qquad N_{1,\,x-2}$$

We find $v_{11} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\operatorname{Ker} p_1(T_A) = N(A - 2I_4)$ and $m_{T_A,\,v_{11}} = m_{T_A,\,v_{12}} = x - 2$. Also

$$\operatorname{Ker}(p_1^{b_1}(T_A)) = N(p_1(A)) = N(A - 2I_4) = C_{T_A,\,v_{11}} \oplus C_{T_A,\,v_{12}}.$$

Note that $C_{T_A,\,v_{11}}$ and $C_{T_A,\,v_{12}}$ have Jordan bases $\beta_{11} : v_{11}$ and $\beta_{12} : v_{12}$ respectively.

CASE 2, $p_2 = x - 3$:

$$p_2(A) = A - 3I_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 4 & 0 & 1 & -1 \end{bmatrix} \quad\rightarrow\quad \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $\nu(p_2(A)) = 1 = \gamma_2$; also $\nu(p_2^2(A)) = 2$. Hence $b_2 = 2$ and we get a corresponding dot diagram consisting of two vertical dots, with associated Jordan block $J_2(3)$:

$$\begin{array}{|c|} \hline \cdot \\ \hline \cdot \\ \hline \end{array} \qquad \begin{array}{l} N_{2,\,x-3} \\ N_{1,\,x-3} \end{array}$$

We have to find a basis of the form $p_2(T_A)(v_{21}) = (A-3I_4)v_{21}$ for $\operatorname{Ker} p_2(T_A) = N(A-3I_4)$.

To find v_{21} we first get a basis for $N(A-3I_4)^2$. We have

$$p_2^2(A) = (A-3I_4)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -10 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we find $X_1 = \begin{bmatrix} 2 \\ 10 \\ 1 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ is such a basis. Then we have

$$\begin{aligned} N_{2,\,p_2} &= \langle p_2X_1,\, p_2X_2 \rangle \\ &= \langle p_2(A)X_1,\, p_2(A)X_2 \rangle = \langle (A-3I_4)X_1,\, (A-3I_4)X_2 \rangle \\ &= \left\langle \begin{bmatrix} 3 \\ -3 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 3 \\ -3 \\ -3 \\ 9 \end{bmatrix} \right\rangle. \end{aligned}$$

Hence we can take $v_{21} = X_1$. Then $m_{T_A,\,v_{21}} = (x-3)^2$. Also

$$\operatorname{Ker} p_2^{b_2}(T_A)) = N(p_2^2(A)) = N(A-3I_4)^2 = C_{T_A,\,v_{21}}.$$

Moreover $C_{T_A,\,v_{21}}$ has Jordan basis $\beta_{21} : v_{21},\, (A-3I_4)v_{21}$.

Finally we have $V_4(\mathbb{Q}) = C_{T_A,\,v_{11}} \oplus C_{T_A,\,v_{12}} \oplus C_{T_A,\,v_{21}}$ and $\beta = \beta_{11} \cup \beta_{12} \cup \beta_{21}$ is a basis for $V_4(\mathbb{Q})$. Then with

$$P = [v_{11}|v_{12}|v_{21}|(A-3I_4)v_{21}] = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 1 & 0 & 10 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 9 \end{bmatrix}$$

we have

$$P^{-1}AP = [T_A]_\beta^\beta = J_1(2) \oplus J_1(2) \oplus J_2(3) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

4.2.2 Example (b):

Let $A \in M_{6\times 6}(F)$ have the property that $ch_A = x^6,\ m_A = x^3$ and

$$\nu(A) = 3,\ \nu(A^2) = 5,\ (\nu(A^3) = 6).$$

Next, with $\nu_{h,\,x} = \dim_F N_{h,\,x}$ we have

$$\begin{aligned}
\nu_{1,\,x} &= \nu(A) = 3 = \gamma_1;\\
\nu_{2,\,x} &= \nu(A^2) - \nu(A) = 5 - 3 = 2;\\
\nu_{3,\,x} &= \nu(A^3) - \nu(A^2) = 6 - 5 = 1.
\end{aligned}$$

Hence the dot diagram corresponding to the (only) monic irreducible factor x of m_A is

·			$N_{3,x}$
·	·		$N_{2,x}$
·	·	·	$N_{1,x}$

Hence we read off that $\exists$ a non-singular $P \in M_{6\times 6}(F)$ such that $P^{-1}AP = J_3(0) \oplus J_2(0) \oplus J_1(0)$. To find such a matrix P we proceed as follows:

(i) First find a basis for $N_{3,\,x}$. We do this by first finding a basis for $N(A^3)$: $X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ X_6$. Then

$$N_{3,\,x} = \langle A^2X_1,\ A^2X_2,\ A^2X_3,\ A^2X_4,\ A^2X_5,\ A^2X_6\rangle.$$

We now apply the LRA (left–to–right algorithm) to the above spanning family to get a basis A^2v_{11} for $N_{3,\,x}$, where A^2v_{11} is the first non–zero vector in the spanning family.

(ii) Now extend the linearly independent family A^2v_{11} to a basis for $N_{2,\,x}$. We do this by first finding a basis $Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5$ for $N(A^2)$. Then

$$N_{2,\,x} = \langle AY_1,\ AY_2,\ AY_3,\ AY_4,\ AY_5\rangle.$$

We now attach A^2v_{11} to the head of this spanning family:

$$N_{2,\,x} = \langle A^2v_{11},\ AY_1,\ AY_2,\ AY_3,\ AY_4,\ AY_5\rangle$$

and apply the LRA to find a basis for $N_{2,\,x}$ which includes A^2X_1. This will have the form $A^2v_{11},\ Av_{12}$, where Av_{12} is the first vector in the list $AY_1, \ldots, AY_5$ which is not a linear combination of A^2v_{11}.

(iii) Now extend the linearly independent family $A^2v_{11},\ Av_{12}$ to a basis for $N_{1,\,x} = N(A)$. We do this by first finding a basis $Z_1,\ Z_2,\ Z_3$ for $N(A)$.

Then place the linearly independent family $A^2v_{11},\, Av_{12}$ at the head of this spanning family:

$$N_{1,\,x} = \langle A^2v_{11},\, Av_{12},\, Z_1,\, Z_2,\, Z_3\rangle.$$

The LRA is then applies to the above spanning family selects a basis of the form $A^2v_{11},\, Av_{12},\, v_{13}$, where v_{13} is the first vector among $Z_1,\, Z_2,\, Z_3$ which is not a linear combination of A^2v_{11} and Av_{12}.

Then $m_{T_A,\,v_{11}} = x^3,\; m_{T_A,\,v_{12}} = x^2,\; m_{T_A,\,v_{13}} = x$. Also

$$\operatorname{Ker} p_1^{b_1}(T_A) = N(A^3) = C_{T_A,\,v_{11}} \oplus C_{T_A,\,v_{12}} \oplus C_{T_A,\,v_{13}}.$$

Finally, if we take Jordan bases

$$\begin{aligned} \beta_{11} &: v_{11},\, Av_{11},\, A^2v_{11}; \\ \beta_{12} &: v_{12},\, Av_{12}; \\ \beta_{13} &: v_{13} \end{aligned}$$

for the three T–cyclic subspaces $C_{T_A,\,v_{11}},\, C_{T_A,\,v_{12}},\, C_{T_A,\,v_{13}}$, respectively, we then get the basis

$$\begin{aligned} \beta &= \beta_{11} \cup \beta_{12} \cup \beta_{13} \\ &= v_{11},\, Av_{11},\, A^2v_{11};\, v_{12},\, Av_{12};\, v_{13} \end{aligned}$$

for $V_6(F)$. Then if

$$P = [v_{11}|Av_{11}|A^2v_{11}|v_{12}|Av_{12}|v_{13}]$$

we have

$$\begin{aligned} P^{-1}AP = [T_A]_\beta^\beta &= J_3(0) \oplus J_2(0) \oplus J_1(0) \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$