4 The Jordan Canonical Form

The following subspaces are central for our treatment of the Jordan and rational canonical forms of a linear transformation $T : V \to V$.

**DEFINITION 4.1**
With $m_T = p_i^{b_i_1} \cdots p_i^{b_i_t}$ as before and $p = p_i$, $b = b_i$ for brevity, we define

$$N_{h,p} = \text{Im} p^{h-1}(T) \cap \text{Ker} p(T).$$

**REMARK.** In numerical examples, we will need to find a spanning family for $N_{h,p}$. This is provided by Problem Sheet 1, Question 11(a): we saw that if $T : U \to V$ and $S : V \to W$ are linear transformations, then if

$$\text{Ker} p^h(T) = \langle u_1, \ldots, u_n \rangle,$$

we have that $N_{h,p} = \langle p^{h-1}u_1, \ldots, p^{h-1}u_n \rangle$.

Hence

$$\dim(\text{Im} T \cap \text{Ker} S) = \nu(ST) - \nu(T).$$

**THEOREM 4.1**

$$N_{1,p} \supseteq N_{2,p} \supseteq \cdots \supseteq N_{b,p} \neq \{0\} = N_{b+1,p} = \cdots.$$

**PROOF.** Successive containment follows from

$$\text{Im} L^{h-1} \supseteq \text{Im} L^h$$

with $L = p(T)$.

The fact that $N_{b,p} \neq \{0\}$ and that $N_{b+1,p} = \{0\}$ follows directly from the formula

$$\dim N_{h,p} = \nu(p^h(T)) - \nu(p^{h-1}(T)).$$

For simplicity, assume that $p$ is linear, that is that $p = x - c$. The general story (when $\deg p > 1$) is similar, but more complicated; it is delayed until the next section.

Telescopic cancellation then gives
**THEOREM 4.2**

\[ \nu_{1,p} + \nu_{2,p} + \cdots + \nu_{b,p} = \nu(p^b(T)) = a, \]

where \( p^a \) is the exact power of \( p \) dividing \( \text{ch}_T \).

Consequently we have the decreasing sequence

\[ \nu_{1,p} \geq \nu_{2,p} \geq \cdots \geq \nu_{b,p} \geq 1. \]

**EXAMPLE 4.1**

Suppose \( T : V \rightarrow V \) is a LT such that \( p^4|m_T, \ p = x - c \) and

\[
\begin{align*}
\nu(p(T)) &= 3, & \nu(p^2(T)) &= 6, \\
\nu(p^3(T)) &= 8, & \nu(p^4(T)) &= 10.
\end{align*}
\]

So

\[
\text{Ker } p(T) \subset \text{Ker } p^2(T) \subset \text{Ker } p^3(T) \subset \text{Ker } p^4(T) = \text{Ker } p^5(T) = \cdots.
\]

Then

\[
\begin{align*}
\nu_{1,p} &= 3, & \nu_{2,p} &= 6 - 3 = 3, \\
\nu_{3,p} &= 8 - 6 = 2, & \nu_{4,p} &= 10 - 8 = 2
\end{align*}
\]

so

\[ N_{1,p} = N_{2,p} \supset N_{3,p} = N_{4,p} \neq \{0\}. \]

**4.1 The Matthews’ dot diagram**

We would represent the previous example as follows:

```
· · · ν_{4,p}  Dots represent dimension:
· · · ν_{3,p}
· · · ν_{2,p}  3 + 3 + 2 + 2 = 10
· · · · · · ν_{1,p}  \Rightarrow 10 = 4 + 4 + 2
```

The conjugate partition of 10 is 4+4+2 (sum of column heights of diagram), and this will soon tell us that there is a corresponding contribution to the **Jordan canonical form** of this transformation, namely

\[ J_4(c) \oplus J_4(c) \oplus J_2(c). \]
In general, and we label the conjugate partition by

\[ e_1 \geq e_2 \geq \cdots \geq e_{\gamma}. \]

Finally, note that the total number of dots in the dot diagram is \( \nu(p^b(T)) \), by Theorem 4.2.

**THEOREM 4.3**

\[ \exists v_1, \ldots, v_{\gamma} \in V \text{ such that} \]

\[ p^{e_1-1}v_1, p^{e_2-1}v_2, \ldots, p^{e_{\gamma}-1}v_{\gamma} \]

form a basis for \( \text{Ker } p(T) \).

**PROOF.** Special case, but the construction is quite general.

Choose a basis \( p^3v_1, p^3v_2 \) for \( N_{1,p} \)

\[ \vdots \]

extend to a basis \( p^3v_1, p^3v_2, pv_3 \) for \( N_{2,p} \)

Then \( p^3v_1, p^3v_2, pv_3 \) is a basis for \( N_{1,p} = \text{Ker } p(T) \).

**THEOREM 4.4 (Secondary decomposition)**

(i) \[ m_{T,v_i} = p^{e_i} \]

(ii) \[ \text{Ker } p^b(T) = C_{T,v_1} \oplus \cdots \oplus C_{T,v_{\gamma}} \]

**PROOF.**
(i) We have $p^{e_i-1}v_i \in \text{Ker} \, p(T)$, so $p^{e_i}v_i = 0$ and hence $m_{T,v_i} = p^f$, where $0 \leq f \leq e_i$.

But $p^{e_i-1}v_i \neq 0$, as it is part of a basis. Hence $f \geq e_i$ and $f = e_i$ as required.

(ii) (a) $C_{T,v_i} \subseteq \text{Ker} \, p^b(T)$.

For $p^{e_i}v_i = 0$ and so $p^{e_i}(fv_i) = 0 \quad \forall f \in F[x]$. Hence as $e_i \leq b$, we have

$$p^b(fv_i) = p^{b-e_i}(p^{e_i}fv_i) = p^{b-e_i}0 = 0$$

and $fv_i \in \text{Ker} \, p^b(T)$. Consequently $C_{T,v_i} \subseteq \text{Ker} \, p^b(T)$ and hence

$$C_{T,v_i} + \cdots + C_{T,v_{\gamma}} \subseteq \text{Ker} \, p^b(T).$$

(b) We presently show that the subspaces $C_{T,v_j}$, $j = 1, \ldots, \gamma$ are independent, so

$$\dim(C_{T,v_1} + \cdots + C_{T,v_{\gamma}}) = \sum_{j=1}^{\gamma} \dim C_{T,v_j}$$

$$= \sum_{j=1}^{\gamma} \deg m_{T,v_j} = \sum_{j=1}^{\gamma} e_j$$

$$= \nu(p^b(T))$$

$$= \dim \text{Ker} \, p^b(T).$$

Hence

$$\text{Ker} \, p^b(T) = C_{T,v_1} + \cdots + C_{T,v_{\gamma}}$$

$$= C_{T,v_1} \oplus \cdots \oplus C_{T,v_{\gamma}}.$$

The independence of the $C_{T,v_i}$ is stated as a lemma:

**Lemma:** Let $v_1, \ldots, v_{\gamma} \in V$, $e_1 \geq \cdots \geq e_\gamma \geq 1$;

$m_{T,v_j} = p^{e_j} \quad 1 \leq j \leq \gamma$; $p = x - c$;

Also $p^{e_{j-1}}v_1, \ldots, p^{e_1-1}v_{\gamma}$ are LI. Then

$$f_1 v_1 + \cdots + f_\gamma v_\gamma = 0 ; \quad f_1, \ldots, f_\gamma \in F[x]$$

$$\Rightarrow \quad p^{e_j} \mid f_j \quad 1 \leq j \leq \gamma.$$
Firstly, consider $e_1 = 1$. Then

$$e_1 = e_2 = \cdots = e_\gamma = 1.$$  

Now $m_{T,v_j} = p^{e_j}$ and

$$p^{e_1-1}v_1, \ldots, p^{e_\gamma-1}v_\gamma$$

are LI, so $v_1, \ldots, v_\gamma$ are LI. So assume

$$f_1v_1 + \cdots + f_\gamma v_\gamma = 0 \quad f_1, \ldots, f_\gamma \in F[x]. \quad (7)$$

and by the remainder theorem

$$f_j = (x-c)q_j + f_j(c). \quad (8)$$

Thus

$$f_jv_j = q_j(x-c)v_j + f_j(c)v_j$$

$$= f_j(c)v_j.$$  

So (7) implies

$$f_1(c)v_1 + \cdots + f_\gamma(c)v_\gamma(c) = 0$$

$$\Rightarrow f_j(c) = 0 \quad \forall j = 1, \ldots, \gamma$$

and (8) implies

$$(x-c) \mid f_j \quad \forall j$$

which is the result.

Now let $e_1 > 1$ and assume the lemma is true for $e_1 - 1$. If

$$m_{T,v_j} = p^{e_j};$$

$$p^{e_1-1}v_1, \ldots, p^{e_\gamma-1}v_\gamma$$

are LI, and

$$f_1v_1 + \cdots + f_\gamma v_\gamma = 0 \quad (9)$$

as before, we have

$$f_1(pv_1) + \cdots + f_\gamma(pv_\gamma) = 0 \quad (10)$$

where $m_{T,pv_j} = p^{e_j-1}$.

Now let $\delta$ be the greatest positive integer such that $e_\delta > 1$; i.e. $e_{\delta+1} = 1$, but $e_\delta > 1$. Applying the induction hypothesis to (10), in the form

$$f_1(pv_1) + \cdots + f_\delta(pv_\delta) = 0$$
we obtain
\[ p^e_j - 1 \mid f_j \quad \forall j = 1, \ldots, \delta, \]
so we may write
\[ f_j = p^{e_j - 1} g_j, \]
(where if \( g_j = f_j \) if \( j > \delta \)). Now substituting in (9),
\[ g_1 p^{e_1 - 1} v_1 + \cdots + g_\gamma p^{e_\gamma - 1} v_\gamma = 0. \]
(11)
But
\[ m_{T, p^{e_j - 1} v_j} = p \]
so (11) and the case \( e_1 = 1 \) give
\[ p \mid g_j \quad \forall j, \]
as required.

**A summary:**

If \( m_T = (x - c_1)^{b_1} \cdots (x - c_t)^{b_t} = p_1^{b_1} \cdots p_t^{b_t} \), then there exist vectors \( v_{ij} \) and positive integers \( e_{ij} \) (1 \( \leq i \leq t, 1 \leq j \leq \gamma_i \)), where \( \gamma_i = \nu(T - c_i I_V) \), satisfying
\[ b_i = e_{i1} \geq \cdots \geq e_{i\gamma_i}, \quad m_{T, v_{ij}} = p_i^{e_{ij}} \]
and
\[ V = \bigoplus_{i=1}^t \bigoplus_{j=1}^{\gamma_i} C_{T, v_{ij}}. \]

We choose the elementary Jordan bases
\[ \beta_{ij} : v_{ij}, (T - c_i I_V)(v_{ij}), \ldots, (T - c_i I_V)^{e_{ij} - 1}(v_{ij}) \]
for \( C_{T, v_{ij}} \). Then if
\[ \beta = \bigcup_{i=1}^t \bigcup_{j=1}^{\gamma_i} \beta_{ij}, \]
\( \beta \) is a basis for \( V \) and we have
\[ [T]_\beta^\gamma = \bigoplus_{i=1}^t \bigoplus_{j=1}^{\gamma_i} J_{e_{ij}}(c_i) = J. \]
A direct sum of elementary Jordan matrices such as \( J \) is called a Jordan canonical form of \( T \).

If \( T = T_A \) and \( P = [v_{11}], \ldots, [v_{t\gamma_t}] \), then
\[ P^{-1} A P = J \]
and \( J \) is called a Jordan canonical form of \( A \).