4 The Jordan Canonical Form

The following subspaces are central for our treatment of the Jordan and rational canonical forms of a linear transformation $T : V \to V$.

DEFINITION 4.1

With $m_T = p_1^{b_1} \dots p_t^{b_t}$ as before and $p = p_i$, $b = b_i$ for brevity, we define

$$N_{h,p} = \operatorname{Im} p^{h-1}(T) \cap \operatorname{Ker} p(T).$$

REMARK. In numerical examples, we will need to find a spanning family for $N_{h,\,p}$. This is provided by Problem Sheet 1, Question 11(a): we saw that if $T : U \to V$ and $S : V \to W$ are linear transformations, then If $\operatorname{Ker} p^h(T) = \langle u_1, \dots, u_n \rangle$, then

$$N_{h,\,p} = \langle p^{h-1}u_1, \dots, p^{h-1}u_n \rangle,$$

where we have taken $U = V = W$ and replaced S and T by $p(T)$ and $p^{h-1}(T)$ respectively, so that $ST = p^h(T)$. Also

$$\dim(\operatorname{Im} T \cap \operatorname{Ker} S) = \nu(ST) - \nu(T).$$

Hence

$$\begin{aligned}
\nu_{h,p} &= \dim N_{h,p} \\
&= \dim(\operatorname{Im} p^{h-1}(T) \cap \operatorname{Ker} p(T)) \\
&= \nu(p^h(T)) - \nu(p^{h-1}(T)).
\end{aligned}$$

THEOREM 4.1

$$N_{1,p} \supseteq N_{2,p} \supseteq \cdots \supseteq N_{b,p} \neq \{0\} = N_{b+1,p} = \cdots.$$

PROOF. Successive containment follows from

$$\operatorname{Im} L^{h-1} \supseteq \operatorname{Im} L^h$$

with $L = p(T)$.

The fact that $N_{b,p} \neq \{0\}$ and that $N_{b+1,p} = \{0\}$ follows directly from the formula

$$\dim N_{h,p} = \nu(p^h(T)) - \nu(p^{h-1}(T)).$$

For simplicity, assume that p is linear, that is that $p = x - c$. The general story (when $\deg p > 1$) is similar, but more complicated; it is delayed until the next section.

Telescopic cancellation then gives

THEOREM 4.2

$$\nu_{1,p} + \nu_{2,p} + \cdots + \nu_{b,p} = \nu(p^b(T)) = a,$$

where p^a is the exact power of p dividing ch_T.

Consequently we have the decreasing sequence

$$\nu_{1,p} \geq \nu_{2,p} \geq \cdots \geq \nu_{b,p} \geq 1.$$

EXAMPLE 4.1

Suppose $T : V \mapsto V$ is a LT such that $p^4 || m_T,\ p = x - c$ and

$$\begin{array}{ll} \nu(p(T)) = 3, & \nu(p^2(T)) = 6, \\ \nu(p^3(T)) = 8, & \nu(p^4(T)) = 10. \end{array}$$

So

$$\operatorname{Ker} p(T) \subset \operatorname{Ker} p^2(T) \subset \operatorname{Ker} p^3(T) \subset \operatorname{Ker} p^4(T) = \operatorname{Ker} p^5(T) = \cdots.$$

Then

$$\begin{array}{ll} \nu_{1,p} = 3, & \nu_{2,p} = 6 - 3 = 3, \\ \nu_{3,p} = 8 - 6 = 2, & \nu_{4,p} = 10 - 8 = 2 \end{array}$$

so

$$N_{1,p} = N_{2,p} \supset N_{3,p} = N_{4,p} \neq \{0\}.$$

4.1 The Matthews' dot diagram

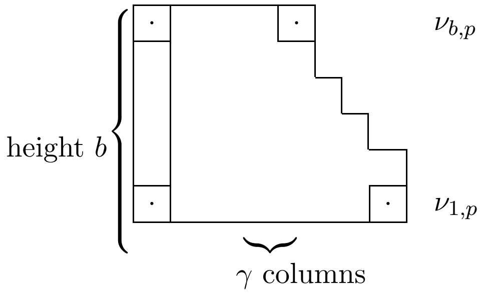
We would represent the previous example as follows:

·	·		$\nu_{4,p}$	Dots represent dimension:
·	·		$\nu_{3,p}$	
·	·	·	$\nu_{2,p}$	$3 + 3 + 2 + 2 = 10$
·	·	·	$\nu_{1,p}$	$\Rightarrow 10 = 4 + 4 + 2$

The conjugate partition of 10 is 4+4+2 (sum of column heights of diagram), and this will soon tell us that there is a corresponding contribution to the **Jordan canonical form** of this transformation, namely

$$J_4(c) \oplus J_4(c) \oplus J_2(c).$$

In general,



and we label the conjugate partition by

$$e_1 \geq e_2 \geq \cdots \geq e_\gamma.$$

Finally, note that the total number of dots in the dot diagram is $\nu(p^b(T))$, by Theorem 4.2.

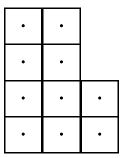
THEOREM 4.3

$\exists v_1, \ldots, v_\gamma \in V$ such that

$$p^{e_1-1}v_1, p^{e_2-1}v_2, \ldots, p^{e_\gamma-1}v_\gamma$$

form a basis for $\operatorname{Ker} p(T)$.

PROOF. Special case, but the construction is quite general.



choose a basis p^3v_1, p^3v_2 for $N_{4,p}$

extend to a basis p^3v_1, p^3v_2, pv_3 for $N_{2,p}$

Then p^3v_1, p^3v_2, pv_3 is a basis for $N_{1,p} = \operatorname{Ker} p(T)$.

THEOREM 4.4 (Secondary decomposition)

(i)

$$m_{T,v_i} = p^{e_i}$$

(ii)

$$\operatorname{Ker} p^b(T) = C_{T,v_1} \oplus \cdots \oplus C_{T,v_\gamma}$$

PROOF.

(i) We have $p^{e_i-1}v_i \in \operatorname{Ker} p(T)$, so $p^{e_i}v_i = 0$ and hence $m_{T,v_i} \mid p^{e_i}$. Hence $m_{T,v_i} = p^f$, where $0 \leq f \leq e_i$.

But $p^{e_i-1}v_i \neq 0$, as it is part of a basis. Hence $f \geq e_i$ and $f = e_i$ as required.

(ii) (a)

$$C_{T,v_i} \subseteq \operatorname{Ker} p^b(T).$$

For $p^{e_i}v_i = 0$ and so $p^{e_i}(fv_i) = 0 \quad \forall f \in F[x]$. Hence as $e_i \leq b$, we have

$$p^b(fv_i) = p^{b-e_i}(p^{e_i}fv_i) = p^{b-e_i}0 = 0$$

and $fv_i \in \operatorname{Ker} p^b(T)$. Consequently $C_{T,v_i} \subseteq \operatorname{Ker} p^b(T)$ and hence

$$C_{T,v_1} + \cdots + C_{T,v_\gamma} \subseteq \operatorname{Ker} p^b(T).$$

(b) We presently show that the subspaces C_{T,v_j}, $j = 1, \ldots, \gamma$ are independent, so

$$\begin{aligned}
\dim(C_{T,v_1} + \cdots + C_{T,v_\gamma}) &= \sum_{j=1}^{\gamma} \dim C_{T,v_j} \\
&= \sum_{j=1}^{\gamma} \deg m_{T,\,v_j} = \sum_{j=1}^{\gamma} e_j \\
&= \nu(p^b(T)) \\
&= \dim \operatorname{Ker} p^b(T).
\end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{Ker} p^b(T) &= C_{T,v_1} + \cdots + C_{T,v_\gamma} \\
&= C_{T,v_1} \oplus \cdots \oplus C_{T,v_\gamma}.
\end{aligned}$$

The independence of the C_{T,v_i} is stated as a lemma:
Lemma: Let $v_1, \ldots, v_\gamma \in V$, $e_1 \geq \cdots \geq e_\gamma \geq 1$;
$m_{T,v_j} = p^{e_j} \quad 1 \leq j \leq \gamma$; $p = x - c$;
Also $p^{e_1-1}v_1, \ldots, p^{e_\gamma-1}v_\gamma$ are LI. Then

$$\begin{aligned}
& f_1v_1 + \cdots + f_\gamma v_\gamma = 0\,; \quad f_1, \ldots, f_\gamma \in F[x] \\
\Rightarrow \quad & p^{e_j} \mid f_j \quad 1 \leq j \leq \gamma.
\end{aligned}$$

PROOF: (induction on e_1)

Firstly, consider $e_1 = 1$. Then

$$e_1 = e_2 = \cdots = e_\gamma = 1.$$

Now $m_{T,v_j} = p^{e_j}$ and

$$p^{e_1-1}v_1, \ldots, p^{e_\gamma-1}v_\gamma$$

are LI, so $v_1, \ldots, v_\gamma$ are LI. So assume

$$f_1 v_1 + \cdots + f_\gamma v_\gamma = 0 \qquad f_1, \ldots, f_\gamma \in F[x]. \tag{7}$$

and by the remainder theorem

$$f_j = (x-c)q_j + f_j(c). \tag{8}$$

Thus

$$\begin{aligned} f_j v_j &= q_j(x-c)v_j + f_j(c)v_j \\ &= f_j(c)v_j. \end{aligned}$$

So (7) implies

$$\begin{aligned} f_1(c)v_1 + \cdots + f_\gamma(c)v_\gamma(c) &= 0 \\ \Rightarrow f_j(c) &= 0 \qquad \forall j = 1, \ldots, \gamma \end{aligned}$$

and (8) implies

$$(x-c) \mid f_j \qquad \forall j$$

which is the result.
Now let $e_1 > 1$ and assume the lemma is true for $e_1 - 1$. If

$$\begin{aligned} &m_{T,v_j} = p^{e_j}; \\ &p^{e_1-1}v_1, \ldots, p^{e_\gamma-1}v_\gamma \text{ are LI,} \\ \text{and} \qquad &f_1 v_1 + \cdots + f_\gamma v_\gamma = 0 \end{aligned} \tag{9}$$

as before, we have

$$f_1(pv_1) + \cdots + f_\gamma(pv_\gamma) = 0 \tag{10}$$

where $m_{T,pv_j} = p^{e_j-1}$.
Now let δ be the greatest positive integer such that $e_\delta > 1$; i.e. $e_{\delta+1} = 1$, but $e_\delta > 1$. Applying the induction hypothesis to (10), in the form

$$f_1(pv_1) + \cdots + f_\delta(pv_\delta) = 0$$

we obtain

$$p^{e_j-1} \mid f_j \qquad \forall j = 1, \ldots, \delta,$$

so we may write

$$f_j = p^{e_j-1} g_j,$$

(where if $g_j = f_j$ if $j > \delta$). Now substituting in (9),

$$g_1 p^{e_1-1} v_1 + \cdots + g_\gamma p^{e_\gamma-1} v_\gamma = 0. \tag{11}$$

But

$$m_{T, p^{e_j-1} v_j} = p$$

so (11) and the case $e_1 = 1$ give

$$p \mid g_j \qquad \forall j,$$

as required.

A summary:

If $m_T = (x - c_1)^{b_1} \ldots (x - c_t)^{b_t} = p_1^{b_1} \ldots p_t^{b_t}$, then there exist vectors v_{ij} and positive integers e_{ij} $(1 \le i \le t,\ 1 \le j \le \gamma_i)$, where $\gamma_i = \nu(T - c_i I_V)$, satisfying

$$b_i = e_{i1} \ge \cdots \ge e_{i\gamma_i}, \quad m_{T, v_{ij}} = p_i^{e_{ij}}$$

and

$$V = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{\gamma_i} C_{T, v_{ij}}.$$

We choose the elementary Jordan bases

$$\beta_{ij} : v_{ij}, (T - c_i I_V)(v_{ij}), \ldots, (T - c_i I_V)^{e_{ij}-1}(v_{ij})$$

for $C_{T, v_{ij}}$. Then if

$$\beta = \bigcup_{i=1}^{t} \bigcup_{j=1}^{\gamma_i} \beta_{ij},$$

β is a basis for V and we have

$$[T]_\beta^\beta = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{\gamma_i} J_{e_{ij}}(c_i) = J.$$

A direct sum of elementary Jordan matrices such as J is called a Jordan canonical form of T.

If $T = T_A$ and $P = [v_{11} | \ldots\ldots | v_{t\gamma_t}]$, then

$$P^{-1} A P = J$$

and J is called a Jordan canonical form of A.