TRANSPOSE OF A MATRIX

DEFINITION. (The transpose of a matrix) Let A be an $m \times n$ matrix. Then A^t, the *transpose* of A, is the matrix obtained by interchanging the rows and columns of A. In other words if $A = [a_{ij}]$, then $\left(A^t\right)_{ji} = a_{ij}$. Consequently A^t is $n \times m$. Here are some properties:

1. $\left(A^t\right)^t = A$;

2. $(A \pm B)^t = A^t \pm B^t$ if A and B are $m \times n$;

3. $(sA)^t = sA^t$ if s is a scalar;

4. $(AB)^t = B^t A^t$ if A is $m \times n$ and B is $n \times p$;

5. If A is non–singular, then A^t is also non–singular and

$$\left(A^t\right)^{-1} = \left(A^{-1}\right)^t;$$

6. $X^t X = x_1^2 + \ldots + x_n^2$ if $X = [x_1, \ldots, x_n]^t$ is a column vector.

We prove only the fourth property. First check that both $(AB)^t$ and B^tA^t have the same size ($p \times m$). Moreover, corresponding elements of both matrices are equal. For if $A = [a_{ij}]$ and $B = [b_{jk}]$, we have

$$\begin{aligned} \left((AB)^t\right)_{ki} &= (AB)_{ik} \\ &= \sum_{j=1}^{n} a_{ij} b_{jk} \\ &= \sum_{j=1}^{n} \left(B^t\right)_{kj} \left(A^t\right)_{ji} \\ &= \left(B^t A^t\right)_{ki}. \end{aligned}$$

There are two important classes of matrices that can be defined concisely in terms of the transpose operation.

DEFINITION. (Symmetric matrix) A real matrix A is called *symmetric* if $A^t = A$. In other words A is square ($n \times n$ say) and $a_{ji} = a_{ij}$ for all $1 \leq i \leq n$, $1 \leq j \leq n$. Hence

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

is a general 2×2 symmetric matrix.

DEFINITION. Skew–symmetric matrix) A real matrix A is called *skew–symmetric* if $A^t = -A$. In other words A is square ($n \times n$ say) and $a_{ji} = -a_{ij}$ for all $1 \leq i \leq n$, $1 \leq j \leq n$.

REMARK. Taking $i = j$ in the definition of skew–symmetric matrix gives $a_{ii} = -a_{ii}$ and so $a_{ii} = 0$. Hence

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

is a general 2×2 skew–symmetric matrix.