2. (i) $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_1 \to \frac{1}{2}R_1 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(ii) $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$;

(iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 + R_3 \\ R_3 \to -R_3 \\ R_2 \leftrightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_3 \\ R_3 \to -R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(iv) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 2R_1 \\ R_1 \to \frac{1}{2}R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

3. (a) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & -1 & 8 \\ 1 & -1 & -1 & -8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & -2 & -2 & -10 \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -8 & -2 \end{bmatrix} R_3 \to \frac{-1}{8}R_3 \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 - 4R_3 \\ R_2 \to R_2 + 3R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & \frac{19}{4} \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$.

The augmented matrix has been converted to reduced row–echelon form and we read off the unique solution $x = -3,\ y = \frac{19}{4},\ z = \frac{1}{4}$.

(b) $\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 5R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 8 & -20 & 4 & 59 \end{bmatrix}$

$R_3 \to R_3 + 2R_2 \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

From the last matrix we see that the original system is inconsistent.

(c) $\begin{bmatrix} 3 & -1 & 7 & 0 \\ 2 & -1 & 4 & \frac{1}{2} \\ 1 & -1 & 1 & 1 \\ 6 & -4 & 10 & 3 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & \frac{1}{2} \\ 3 & -1 & 7 & 0 \\ 6 & -4 & 10 & 3 \end{bmatrix}$

$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & \frac{-3}{2} \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_2 \\ R_4 \to R_4 - R_3 \\ R_3 \to R_3 - 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 3 & \frac{-1}{2} \\ 0 & 1 & 2 & \frac{-3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The augmented matrix has been converted to reduced row–echelon form and we read off the complete solution $x = -\frac{1}{2} - 3z,\ y = -\frac{3}{2} - 2z$, with z arbitrary.

4. $\left[\begin{array}{rrrc} 2 & -1 & 3 & a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{array}\right] R_2 \to R_2 - R_1 \left[\begin{array}{rrrc} 2 & -1 & 3 & a \\ 1 & 2 & -8 & b-a \\ -5 & -5 & 21 & c \end{array}\right]$

$$R_1 \leftrightarrow R_2 \left[\begin{array}{rrrc} 1 & 2 & -8 & b-a \\ 2 & -1 & 3 & a \\ -5 & -5 & 21 & c \end{array}\right] \begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 5R_1 \end{array} \left[\begin{array}{rrrc} 1 & 2 & -8 & b-a \\ 0 & -5 & 19 & -2b+3a \\ 0 & 5 & -19 & 5b-5a+c \end{array}\right]$$

$$\begin{array}{c} R_3 \to R_3 + R_2 \\ R_2 \to \frac{-1}{5} R_2 \end{array} \left[\begin{array}{rrrc} 1 & 2 & -8 & b-a \\ 0 & 1 & \frac{-19}{5} & \frac{2b-3a}{5} \\ 0 & 0 & 0 & 3b-2a+c \end{array}\right]$$

$$R_1 \to R_1 - 2R_2 \left[\begin{array}{rrrc} 1 & 0 & \frac{-2}{5} & \frac{(b+a)}{5} \\ 0 & 1 & \frac{-19}{5} & \frac{2b-3a}{5} \\ 0 & 0 & 0 & 3b-2a+c \end{array}\right].$$

From the last matrix we see that the original system is inconsistent if $3b-2a+c \neq 0$. If $3b-2a+c = 0$, the system is consistent and the solution is

$$x = \frac{(b+a)}{5} + \frac{2}{5}z, \ y = \frac{(2b-3a)}{5} + \frac{19}{5}z,$$

where z is arbitrary.

5.

$$\left[\begin{array}{cc} \lambda-3 & 1 \\ 1 & \lambda-3 \end{array}\right] R_1 \leftrightarrow R_2 \left[\begin{array}{cc} 1 & \lambda-3 \\ \lambda-3 & 1 \end{array}\right]$$
$$R_2 \to R_2 - (\lambda-3)R_1 \left[\begin{array}{cc} 1 & \lambda-3 \\ 0 & -\lambda^2+6\lambda-8 \end{array}\right] = B.$$

Case 1: $-\lambda^2+6\lambda-8 \neq 0$. That is $-(\lambda-2)(\lambda-4) \neq 0$ or $\lambda \neq 2, 4$. Here B is row equivalent to $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$:

$$R_2 \to \frac{1}{-\lambda^2+6\lambda-8} R_2 \left[\begin{array}{cc} 1 & \lambda-3 \\ 0 & 1 \end{array}\right] R_1 \to R_1 - (\lambda-3)R_2 \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

Hence we get the trivial solution $x = 0, \ y = 0$.

Case 2: $\lambda = 2$. Then $B = \left[\begin{array}{cc} 1 & -1 \\ 0 & 0 \end{array}\right]$ and the solution is $x = y$, with y arbitrary.

Case 3: $\lambda = 4$. Then $B = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]$ and the solution is $x = -y$, with y arbitrary.

6.

$$\begin{aligned}
\left[\begin{array}{rrrr} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{array}\right] R_1 &\to \frac{1}{3}R_1 \left[\begin{array}{rrrr} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 5 & -1 & 1 & -1 \end{array}\right] \\
R_2 &\to R_2 - 5R_1 \left[\begin{array}{rrrr} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} \end{array}\right] \\
R_2 &\to \frac{-3}{8}R_2 \left[\begin{array}{rrrr} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{4} & 1 \end{array}\right] \\
R_1 &\to R_1 - \frac{1}{3}R_2 \left[\begin{array}{rrrr} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 1 \end{array}\right].
\end{aligned}$$

Hence the solution of the original homogeneous system is

$$x_1 = -\frac{1}{4}x_3,\ x_2 = -\frac{1}{4}x_3 - x_4,$$

with x_3 and x_4 arbitrary.

7. <u>Method 1</u>.

$$\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{array}{c} R_1 \to R_1 - R_4 \\ R_2 \to R_2 - R_4 \\ R_3 \to R_3 - R_4 \end{array} \begin{bmatrix} -4 & 0 & 0 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & -4 & 4 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & -3 \end{bmatrix} R_4 \to R_4 - R_3 - R_2 - R_1 \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the given homogeneous system has complete solution

$$x_1 = x_4,\ x_2 = x_4,\ x_3 = x_4,$$

with x_4 arbitrary.

<u>Method 2</u>. Write the system as

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4x_1 \\ x_1 + x_2 + x_3 + x_4 &= 4x_2 \\ x_1 + x_2 + x_3 + x_4 &= 4x_3 \\ x_1 + x_2 + x_3 + x_4 &= 4x_4. \end{aligned}$$

Then it is immediate that any solution must satisfy $x_1 = x_2 = x_3 = x_4$. Conversely, if x_1, x_2, x_3, x_4 satisfy $x_1 = x_2 = x_3 = x_4$, we get a solution.

8.

$$A = \begin{bmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix} \begin{array}{c} R_1 \to R_1 - R_n \\ R_2 \to R_2 - R_n \\ \vdots \\ R_{n-1} \to R_{n-1} - R_n \end{array} \begin{bmatrix} -n & 0 & \cdots & n \\ 0 & -n & \cdots & n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix} R_n \to R_n - R_{n-1} \cdots - R_1 \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.

Consequently the homogeneous system with coefficient matrix A has the solution

$$x_1 = x_n,\ x_2 = x_n, \ldots, x_{n-1} = x_n,$$

with x_n arbitrary.

Alternatively, writing the system in the form

$$\begin{aligned} x_1 + \cdots + x_n &= nx_1 \\ x_1 + \cdots + x_n &= nx_2 \\ &\vdots \\ x_1 + \cdots + x_n &= nx_n \end{aligned}$$

shows that any solution must satisfy $nx_1 = nx_2 = \cdots = nx_n$, so $x_1 = x_2 = \cdots = x_n$. Conversely if $x_1 = x_n, \ldots, x_{n-1} = x_n$, we see that $x_1, \ldots, x_n$ is a solution.

9. (i) Suppose that $A^2 = 0$. Then if A^{-1} exists, we deduce that $A^{-1}(AA) = A^{-1}0$, which gives $A = 0$ and this is a contradiction, as the zero matrix is singular. We conclude that A does not have an inverse.

(ii). Suppose that $A^2 = A$ and that A^{-1} exists. Then

$$A^{-1}(AA) = A^{-1}A,$$

which gives $A = I_n$. Equivalently, if $A^2 = A$ and $A \neq I_n$, then A does not have an inverse.

11.

$$\begin{aligned} A &= E_3(2)E_{14}E_{42}(3) = E_3(2)E_{14}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \\ &= E_3(2)\begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Also

$$\begin{aligned} A^{-1} &= (E_3(2)E_{14}E_{42}(3))^{-1} \\ &= (E_{42}(3))^{-1}E_{14}^{-1}(E_3(2))^{-1} \\ &= E_{42}(-3)E_{14}E_3(1/2) \\ &= E_{42}(-3)E_{14}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= E_{42}(-3)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}. \end{aligned}$$

12. If A is $m \times m$ and is row–equivalent to a matrix having a zero row, then rrref(A) will have its last row zero and hence the homogeneous system $AX = 0$ will have a non–trivial solution. Consequently A cannot be non–singular.

13.

(a) $$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_3 \to \frac{1}{2}R_3 \\ R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3 \\ R_1 \leftrightarrow R_3 \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 1/2 \\ 0 & 1 & 1 & 1 & 0 & -1/2 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right].$$

Hence A^{-1} exists and

$$A^{-1} = \left[\begin{array}{ccc} 0 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1 & -1 & -1 \end{array}\right].$$

(b) $$\left[\begin{array}{ccc|ccc} 2 & 2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \to R_1 - 2R_2 \\ R_1 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & -2 & -2 \end{array}\right]$$

$$R_3 \to \tfrac{1}{2}R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1 & -1 \end{array}\right]$$

$$R_1 \to R_1 - R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1 & -1 \end{array}\right].$$

Hence A^{-1} exists and

$$A^{-1} = \left[\begin{array}{ccc} -1/2 & 2 & 1 \\ 0 & 0 & 1 \\ 1/2 & -1 & -1 \end{array}\right].$$

(c) $$\left[\begin{array}{ccc} 4 & 6 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{array}\right] \begin{array}{c} R_2 \to \frac{1}{7}R_2 \\ R_3 \to \frac{1}{5}R_3 \end{array} \left[\begin{array}{ccc} 4 & 6 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2 \quad \left[\begin{array}{ccc} 4 & 6 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

Hence A is singular by virtue of the zero row.

(d) $$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \to \frac{1}{2}R_1 \\ R_2 \to \frac{-1}{5}R_2 \\ R_3 \to \frac{1}{7}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/7 \end{array}\right].$$

Hence A^{-1} exists and $A^{-1} = \mathrm{diag}\,(1/2,\, -1/5,\, 1/7)$.

(e) $$\left[\begin{array}{cccc|cccc} 1 & 2 & 4 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 - 2R_2 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 6 & 1 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 2R_3 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 6 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 \to R_1 - 3R_4 \\ R_2 \to R_2 + 2R_4 \\ R_3 \to R_3 - R_4 \\ R_4 \to \frac{1}{2}R_4 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right].$$

Hence A^{-1} exists and

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

(f)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix} \begin{array}{c} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 5R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{bmatrix} \; R_3 \to R_3 - R_2 \; \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence A is singular by virtue of the zero row.

14. (i) Let A be $m \times n$ and B be $n \times m$, where $m > n$. Then the homogeneous system $BX = 0$ has a non–trivial solution X_0, as the number of unknowns is greater than the number of equations. Then

$$(AB)X_0 = A(BX_0) = A0 = 0$$

and the $m \times m$ matrix AB is therefore singular, as $X_0 \neq 0$.

(ii) (a) Let B be a singular $n \times n$ matrix. Then $BX = 0$ for some non–zero column vector X. Then $(AB)X = A(BX) = A0 = 0$ and hence AB is also singular.

(b) Suppose A is a singular $n \times n$ matrix. Then A^t is also singular and hence by (i) so is $B^tA^t = (AB)^t$. Consequently AB is also singular.

15. Let A be $m \times m$. then

(i) Suppose $AX = 0 \Rightarrow X = 0$ for all $X \in \mathbb{R}^m$. Then clearly rref(A) must be I_m and hence A is non–singular.

(ii) Suppose B is $m \times m$ and $BA = I_m$.

(iii) We first show that A is non–singular. Assume $AX = 0$. Then $B(AX) = B0 = 0$, so $(BA)X = 0$, $I_mX = 0$ and hence $X = 0$.

Then from $BA = I_n$ we deduce $(BA)A^{-1} = I_mA^{-1}$ and hence $B = A^{-1}$. The equation $AA^{-1} = I_m$ then gives $AB = I_m$.

(iii) Suppose A is non–singular and let rref$(A) = B$. Then B cannot have a zero row and this implies that $B = I_m$.

16. Let $A = I_n - B$, where $B^t = -B$. To prove A is non–singular, it suffices to show that $AX = 0$ implies $X = 0$. So assume $AX = 0$. Then $(I_n - B)X = 0$, so $X = BX$. Hence $X^tX = X^tBX$.

Taking transposes of both sides gives

$$\begin{aligned} (X^tBX)^t &= (X^tX)^t \\ X^tB^t(X^t)^t &= X^t(X^t)^t \\ X^t(-B)X &= X^tX \\ -X^tBX &= X^tX = X^tBX. \end{aligned}$$

Hence $X^tX = -X^tX$ and $X^tX = 0$. But if $X = [x_1, \ldots, x_n]^t$, then $X^tX = x_1^2 + \ldots + x_n^2 = 0$ and hence $x_1 = 0, \ldots, x_n = 0$.