REVISION PROBLEM SHEET 1, MP204/274, Semester 1, 1999

1. Let $A = [1, 1, 1]$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$. Find the entry in the first row, second column of AB^t. (Ans: 5.)

2. Find the values of k for which the matrix $A = \begin{bmatrix} 1 & 1 & k \\ 1 & 1 & 1 \\ k & 1 & 1 \end{bmatrix}$ has rank (a) 1, (b) 2, (c) 3. (Ans: (a) $k = 1$, (b) none (c) $k \neq 1$.)

3. Find the first row of the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. (Ans $[1, 2, -2]$.)

4. For what values of k does the vector $(2, 1, k)$ lie in the subspace spanned by $(1, 1, 1)$ and $(1, 0, 2)$? (Ans: $k = 3$.)

5. Which of the following are subspaces of $\mathbb{R}^3$?

   (a) $\{(x, y, z)|2x + y = 2z\}$;
   (b) $\{(x, y, z)|x^2 + y^2 + z^2 = 0\}$;
   (c) $\{(x, y, z)|x^3 + y^3 + z^3 = 0\}$.

   (Ans: (a) and (b).)

6. Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 2 & -2 & 1 & 2 & 3 \\ -2 & 2 & -7 & -2 & 3 \end{bmatrix}$.

   Show that $(1, 2, -2)^t$ and $(2, 1, -7)^t$ form a basis for $C(A)$. Also which of the following are also bases for $C(A)$?

   (a) $(1, 2, -2)^t, (2, 1, -7)^t, (0, 3, 3)^t$;
   (b) $(1, 0, 0)^t, (0, 3, 3)^t$.
   (c) $(1, 5, 1)^t, (0, 3, 3)^t$.

   (Ans: (c).)

7. Find the values of k such that the following are linearly independent:

   $(1, -1, 0, 0), (1, 1, 1, 0), (1, 0, 0, 1), (k, 1, 1, 2k)$.

   (Ans: $k \neq -1$.)

8. The set $W = \{(x_1, x_2, x_3, x_4)|x_2 = x_4$ and $x_1 = x_3\}$ is a subspacae of $\mathbb{R}^4$. Find a basis for W. (One answer is $(0, 1, 0, 1)$, $(1, 1, 1, 1)$.)

9. $\mathbb{R}^2$ becomes a vector space under the operations

   (a) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1,\ x_2 + y_2 + 1)$;
   (b) $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2 + \lambda - 1)$.

   If $Z = -(1, 2)$, find $2Z$. (Ans: $(-2, -7)$.)

10. Answer the following with true (T) or false (F):

(a) A system of m linear equations in n unknowns with $m > n$ is always inconsistent. (F)

(b) If A is symmetric, so is $B^t AB$. (T)

(c) The sum of two elementary row matrices is again an elementary row matrix. (F)

(d) Every vector space with at least two elements has at least two subspaces. (T)

(e) Vectors $(1,0,1),\ (0,1,0),\ (-1,1,-1)$ span $\mathbb{R}^3$. (F)

(f) If vectors v_1, v_2, v_3 are linearly independent in a vector space V, then $\langle v_1, v_2\rangle = \langle v_2, v_3\rangle$. (F)

11. Prove that if A is a square matrix, then $\operatorname{rank} A \geq \operatorname{rank} A^2$. Also give an example of a 2×2 matrix A for which $\operatorname{rank} A > \operatorname{rank} A^2$ holds.

12. If A and B are invertible symmetric matrices such that $AB = BA$, prove that $A^{-1}B$ is also an invertible symmetric matrix.

13. Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 0 & 3 \\ 0 & 1 & -4 & 1 \end{bmatrix}$. Find non–singular matrices P and Q such that $PAQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.