Quotient spaces

V is a vector space and W is a subspace of V.

A *left coset* of W in V is a subset of the form $v + W = \{v + w | w \in W\}$.

THEOREM 1. $u + W = v + W \Leftrightarrow u - v \in W$.

PROOF. "$\Rightarrow$": Suppose $u + W = v + W$. Then $u = u + 0$ and $0 \in W$. Hence $u \in u + W$. Hence $u \in v + W$, so $u = v + w, w \in W$. Hence $u - v = w \in W$.

"$\Leftarrow$": Suppose $u - v = w \in W$. Then

$$\begin{aligned} x \in u + W \Rightarrow x &= u + w_1,\ w_1 \in W \\ \Rightarrow x &= (v + w) + w_1 \\ &= v + (w + w_1) \in v + W. \end{aligned}$$

Similarly $x \in v + W \Rightarrow x \in u + W$.

THEOREM 2. Addition and scalar multiplication on V/W can be unambiguously defined by

(i) $(u + W) + (v + W) = (u + v) + W$, if $u, v \in V$;

(ii) $\lambda(v + W) = \lambda v + W$, if $v \in V$ and $\lambda \in F$.

Moreover, the collection V/W of all left cosets (called the quotient space of V by W) is a vector space under these operations, with zero vector W and $-(v + W) = (-v) + W$.

PROOF. (i) Let $u + W = u' + W$ and $v + W = v' + W$. We have to prove We have $u - u' \in W$ and $v - v' \in W$. Hence $(u + u') - (v + v') = (u - u') + (v - v') \in W$ and by Theorem 1, we have $(u + v) + W = (u' + v') + W$.

(ii) Suppose $v + W = v' + W$. We have to show $\lambda v + W = \lambda' + W$, if $\lambda \in \mathbb{R}$. But $v - v' \in W$, so $\lambda(v - v') = \lambda v - \lambda v' \in W$ and by Theorem 1, we have $\lambda v + W = \lambda' + W$.

There are eight axioms to be verified. We only verify two.

(i)$(s+t)(u+W) = (s+t)u+W = (su+tu)+W$

$= (su+W)+(tu+W) = s(u+W)+t(u+W)$.

(ii) $W+(u+W) = (0+W)+(u+W) = (0+u)+W = u+W$. Hence W is the zero vector here.

THEOREM 3. If $w_1, \ldots, w_r$ is a basis for W and $w_1, \ldots, w_r$, $w_{r+1}, \ldots, w_n$ is a basis for V, then $w_{r+1} + W, \ldots, w_n + W$ form a basis for V/W, thus proving the formula

$$\dim V/W = \dim V - \dim W.$$

PROOF. (i) $V/W = \langle w_{r+1} + W, \ldots, w_n + W \rangle$.

For if $v \in V$, then

$v = x_1 w_1 + \ldots + x_r w_r + x_{r+1} w_{r+1} + \cdots + x_n w_n$.

Hence, noting $w + v + W = v + W$, if $w \in W$, we have $v + W =$

$(x_1 w_1 + \ldots + x_r w_r) + (x_{r+1} w_{r+1} + \cdots + x_n w_n) + W$

$$= (x_{r+1} w_{r+1} + \cdots + x_n w_n) + W$$

$$= x_{r+1}(w_{r+1} + W) + \cdots + x_n(w_n + W)$$

(ii) $w_{r+1} + W, \ldots, w_n + W$ are linearly independent. For suppose

$$x_{r+1}(w_{r+1} + W) + \cdots + x_n(w_n + W) = W.$$

Then $x_{r+1}w_{r+1} + \cdots + x_n w_n + W = W$ and

$x_{r+1}w_{r+1} + \cdots + x_n w_n \in W$. Consequently

$x_{r+1}w_{r+1} + \cdots + x_n w_n = x_1 w_1 + \cdots + x_r w_r$ and

$(-x_1)w_1 + \cdots + (-x_r)w_r + x_{r+1}w_{r+1} + \cdots + x_n w_n = 0.$

Hence $x_{r+1} = 0, \ldots, x_n = 0$.