PROBLEM SHEET 9, MP204/274, Semester 1, 1999

1. Let $P_n[\mathbb{R}]$ denote the vector space of polynomials of degree not exceeding n. Let $c_1, c_2, \ldots, c_{n+1}$ be distinct real numbers and define $f(x) \cdot g(x)$ for $f(x)$, $g(x) \in P_n[\mathbb{R}]$ by

$$f(x) \cdot g(x) = \sum_{i=1}^{n+1} f(c_i)g(c_i).$$

Show that this defines an inner product on $P_n[\mathbb{R}]$.

2. Let V be a real inner product space with inner product $u \cdot v$. If $T : V \to V$ is an isomorphism, prove that the function $f : V \times V \to \mathbb{R}$ defined by $f(u, v) = T(u) \cdot T(v)$ is an inner product on V.

3. Let $v_1, \ldots, v_r$ be vectors in an inner product space V. Prove that the Gram matrix $G = [v_i \cdot v_j]$ is singular if and only if $v_1, \ldots, v_r$ are linearly dependent.

4. If $X = (a_1, b_1, c_1)^t$ and $Y = (a_2, b_2, c_2)^t$, we define the *cross–product* $X \times Y = (a, b, c)^t$, where

$$a = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad b = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad c = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

   (i) Show that $X \cdot (X \times Y) = 0 = Y \cdot (X \times Y)$.

   (ii) Show that $X \times X = 0$.

   (iii) Show that $Y \times X = -(X \times Y)$.

   (iv) $X \times (Y + Z) = X \times Y + X \times Z$.

   (v) $(tX) \times Y = t(X \times Y)$.

   (vi) (Scalar triple product formula) If $Z = (a_3, b_3, c_3)^t$, then

$$X \cdot (Y \times Z) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (X \times Y) \cdot Z;$$

   (vii) $||X \times Y|| = \sqrt{||X||^2||Y||^2 - (X \cdot Y)^2}$.

   (viii) If X and Y are non–zero vectors, prove that $X \times Y = 0$ if and only if X and Y are linearly dependent.

   (ix) If X and Y are linearly independent, prove that Z is orthogonal to X and Y if and only if $Z = t(X \times Y)$ for some $t \in \mathbb{R}$.

5. V is a real inner product space and U is a subspace of V. Let $U^\perp$ be the subset of V defined by

$$U^\perp = \{v \in V | v \cdot u = 0, \forall u \in U\}$$

   (a) Show that $U^\perp$ is a subspace of V.

   (b) If $U = \langle X, Y \rangle$, where X and Y are linearly independent vectors in $\mathbb{R}^3$, prove that $U^\perp = \langle X \times Y \rangle$.

(c) Let $u_1, \ldots, u_r$ and $u_1, \ldots, u_r, u_{r+1}, \ldots, u_n$ be orthonormal bases for U and V respectively. Prove that $u_{r+1}, \ldots, u_n$ is an orthonormal basis for $U^\perp$, thereby proving that $V = U \oplus U^\perp$ and

$$\dim U^\perp + \dim U = \dim V.$$

(d) If V is finite–dimensional and U is a subspace of V, prove that $(U^\perp)^\perp = U$. (Hint: Use (c).)

6. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by

$$u_1 = (1,1,1,1)^t, u_2 = (0,1,1,1)^t, u_3 = (0,0,1,1)^t.$$

Extend this to an orthonormal basis for $\mathbb{R}^4$.

7. Find an orthonormal basis for $P_2[\mathbb{R}]$ using the inner product

$$f(x) \cdot g(x) = \int_0^1 f(t)g(t)\,dt$$

by applying the Gram–Schmidt process to the basis 1, x, x^2.
[Answer: 1, $\sqrt{12}(x - \frac{1}{2})$, $\sqrt{180}(x^2 - x + \frac{1}{6})$.]

8. Let

$$A = \begin{bmatrix} 1 & -4 & 0 \\ -4 & 3 & -4 \\ 0 & -4 & 5 \end{bmatrix}.$$

Given that $ch_A(x) = (x-9)(x-3)(x+3)$, find an orthogonal matrix P such that $P^tAP = \mathrm{diag}\,(9, 3, -3)$.

9. Let

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}.$$

Given that $ch_A(x) = (x-3)^2(x-9)$, find an orthogonal matrix P such that $P^tAP = \mathrm{diag}\,(3, 3, 9)$.

10. Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find $ch_A(x)$ and deduce that A is positive definite by showing that the eigenvalues are positive.