PROBLEM SHEET 8, MP204/274, Semester 1, 1999

1. If $A \in M_{n\times n}(\mathbb{R})$, prove that

$$N(A^h) = N(A^{h+1}) \Rightarrow N(A^{h+1}) = N(A^{h+2}).$$

2. Solve the system of differential equations

$$\begin{aligned} \dot{x} &= -4x + 6y \\ \dot{y} &= -x + y, \end{aligned}$$

given that $x = 5$ and $y = 2$ when $t = 0$

[Ans: $x = 2e^{-t} + 3e^{-2t}$, $y = e^{-t} + e^{-2t}$.]

3. Let $A \in M_{3\times 3}(\mathbb{R})$ satify $A^3 = 0$ and $A^2 \neq 0$. If X satifies $A^2X \neq 0$, prove that X, AX, A^2X are linearly independent. Also show that if $P = [X|AX|A^2X]$, then $P^{-1}AP = J_3(0)$. State a generalisation with A replaced by $A - \lambda I_3$.

4. Let $A = \begin{bmatrix} 4 & -7 & 2 \\ 3 & -5 & 1 \\ -1 & 2 & -2 \end{bmatrix} \in M_{3\times 3}(\mathbb{R})$. Verify that $ch_A(x) = (x+1)^3$ and find a block upper triangular form for A (using the algorithm given in lectures) and also its Jordan form, using Q3.

Note: To aid the calculations, $(A + I_3)^2 = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 2 & -3 & 1 \end{bmatrix}$.

Hence prove that

$$A^n = (-1)^n \begin{bmatrix} n^2 - 6n + 1 & \frac{-3n^2+17n}{2} & \frac{n^2-5n}{2} \\ n^2 - 4n & \frac{-3n^2+11n}{2} + 1 & \frac{n^2-3n}{2} \\ n^2 & \frac{-3n^2-n}{2} & \frac{n^2+n}{2} + 1 \end{bmatrix}.$$

5. For the following matrix A, find $ch_A(x)$ and find a non-singular P such that $P^{-1}AP$ is in block upper triangular form:

$$\begin{bmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -4 & 0 & 0 \end{bmatrix}$$

Also find J_A and a non-singular P such that $P^{-1}AP = J_A$.

6. For the following real matrix A, given that $ch_A(x) = (x-2)^5$, find a non-singular P such that $P^{-1}AP$ is in block upper triangular form:

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -4 & 1 & -3 & 2 & 1 \\ -2 & -1 & 0 & 1 & 1 \\ -3 & -1 & -3 & 4 & 1 \\ -8 & -2 & -7 & 5 & 4 \end{bmatrix}.$$

Also find J_A and a non-singular P such that $P^{-1}AP = J_A$.

Note: To aid the calculations, if $B = A - 2I_5$,

$$rref(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{bmatrix}$$

and $B^3 = 0$.

7. Let $A \in M_{3\times 3}(\mathbb{R})$ satisfy $A^2 = 0$ and $A \neq 0$. We know that nullity $A = 2$ from Q1, Sheet 4. Let $AX_1 \neq 0$ and let $X_2 = AX_1$. If X_3 is a vector such that X_2, X_3 form a basis for $N(A)$, show that X_1, X_2, X_3 are linearly independent and that if $P = [X_1|X_2|X_3]$, then $P^{-1}AP = J_2(0) \oplus J_1(0)$. State a generalisation with A replaced by $A - \lambda I_3$.

   Let $A = \begin{bmatrix} 0 & 4 & -3 \\ -1 & -5 & 3 \\ -1 & -4 & 2 \end{bmatrix} \in M_{3\times 3}(\mathbb{R})$.

   Verify that $ch_A(x) = (x+1)^3$ and $(A + I_3)^2 = 0$. Find a non–singular $P \in M_{3\times 3}(\mathbb{R})$ such that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

   Also prove A is not diagonable over $\mathbb{R}$.

8. Let $A \in M_{3\times 3}(\mathbb{R})$, $ch_A(x) = (x-a)^2(x-b)$, where $a \neq b$. Also assume $(A - aI_3)(A - bI_3) \neq 0$. Let $B = A - aI_3$ and $C = A - bI_3$.

   (a) Explain why nullity $B^2 = 2$.

   (b) (∗) Prove that nullity $B = 1$. (Hint: Assume nullity $B = 2$ and use the equation $B^2C = 0$ to deduce the contradiction $BC = 0$.)

   (c) Use Q8 (b), Sheet 5 to show $\dim C(B) \cap N(B) = 1$.

   (d) Let $C(B) \cap N(B) = \langle X_2 \rangle$ and let $X_2 = BX_1$.
   Prove that X_1 and X_2 form a basis for $G_A(a)$.

   (e) Use the identity $B^2 = (A - (b - 2a)I_3)C + (a - b)^2 I_3$ to show directly that the generalised eigenspaces $G_A(a)$ and $G_A(b)$ are independent.

   (f) Let X_3 be a basis for $N(C)$. Use (e) to explain why X_1, X_2, X_3 are linearly independent and prove that if $P = [X_1|X_2|X_3]$, then $P^{-1}AP = J_2(a) \oplus J_1(b)$.

   (g) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & -2 \end{bmatrix}$. Verify $ch_A(x) = (x-3)^2(x+2)$ and find a non–singular P such that $P^{-1}AP = J_2(3) \oplus J_1(-2)$.

9. Matrices A, B, $C \in M_{3\times 3}(\mathbb{R})$ are defined by

$$A = \begin{bmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & -1 \\ -4 & 4 & -2 \\ -2 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}.$$

Given that $ch_A(x) = ch_B(x) = ch_C(x) = (x-2)^2(x-1)$, decide which pairs of matrices are similar over $\mathbb{R}$. (Answer: A and C.)